LECTURE 2. (TEXED): IN CLASS: PROBABLY LECTURE 3. MANIFOLDS 1. FALL 2006. TANGENT VECTORS.

Overview: Tangent vectors, spaces and bundles. First: to an embedded manifold of Euclidean space. Then to one defined by the vanishing of a submersion (the Implicit FT). Then abstract, via curves.
Vector structure on the tangent space.
Vector bundle structure.

Reading: Milnor, p. 2,3. Lee, ch.3.

1. The tangent space to an embedded submanifold.

We begin with the sphere, \( S^2 \subset \mathbb{R}^3 \). Then the tangent space to \( \omega \in S^2 \) consists of the plane orthogonal to \( \omega \) and passing through \( \omega \). Why? What is the definition of tangent space?

We will begin with the following definition for an embedded manifold \( M^n \) of a Euclidean space \( E \). Let \( p \in M \). If \( c \) is a smooth curve in \( M \) passing through \( m \) at time \( t = 0 \), then we can take its derivative at \( t = 0 \): \( dc/dt|_{t=0} \in E \). We assemble all of these together to form the tangent space to \( M \) at \( p \). This space is denoted \( T_pM \).

Definition 1.1.
\[
T_pM = \{ dc/dt|_{t=0} : c \text{ a smooth curve in } M \text{ with } c(0) = p \}
\]
This set is known as the tangent space to \( M \) at \( m \)

Proposition 1.2. If \( f : U \subset \mathbb{R}^n \) is any parameterization of \( M \) covering a nbhd of \( m \) then \( T_pM = \text{im}(df(q)) \) where \( q = f^{-1}(m) \).

Proof. In the definition, the domain of \( c \) does not matter as long as it contains 0 which is mapped to \( p \) under \( c \). So we can assume that the curves \( c \) lie in \( f(U) \). Then any such curve has the form \( f \circ \gamma(t) \) where \( \gamma(t) \) is an arbitrary smooth curve through \( q \) at \( t = 0 \). We then have \( dc/dt|_{t=0} = df \circ v \) where \( v = d\gamma/dt|_{t=0} \). By varying \( \gamma \) we can make any vector \( v \) in \( \mathbb{R}^n \). For example, just set \( \gamma(t) = q + tv \). So the collection of \( dc/dt(0)'s \) coincides with \( \text{im}(df_q) \). QED

This proposition shows that the tangent spaces are endowed with a linear structure, being the image of a linear space under a linear map. But the tangent space has no canonical basis, because there are no canonical parameterization \( f : \mathbb{R}^n \rightarrow M \).

Remark: affine version For the purposes of visualization it is often best, with embedded submanifolds, to translate the tangent space as defined above so as to obtain a plane passing through \( p \). We will call this the affine tangent space, and write it as: \( T_pM^{\text{affine}} := p + T_pM \). This is the definition that best fits with first quarter calculus. For example, a function \( y = f(x) \), as thought of in calculus, can be turned into an embedded 1-manifold of the plane by using its graph: \( x \mapsto (x, f(x)) \).
Proposition 1.3. Suppose that the embedded submanifold $M$ is defined, as per the Implicit Function Theorem, as the level set of a submersion: $M = \{ m : g(m) = A \}$ where $g : \mathbb{R}^N \to \mathbb{R}^k$ and $A \in \mathbb{R}^k$ is a regular value of $g$. Then $T_pM = \ker(dg_p)$.

Proof. The tangent space at $m$ has dimension $n$, and $n + k = N$ by the proof of the implicit function theorem. Now, suppose $c$ is any curve lying in $M$. Then $f(c(t)) = A$ is constant, since $g$ is identically equal to $A$ on $M$. Differentiating, we find that $dg_p(dc/dt(0)) = 0$, so that $dc/dt(0) \in \ker(dg_p)$. Since the $dc/dt$'s make up $T_pM$ we have that $T_pM \subset \ker(dg_p)$. But the dimension of $T_pM$ is $n$, from the proof of the IFT, and the dimension of $\ker(dg_p)$ is also $n$, by the theorem that the dimension of the kernel plus the image equals the dimension of the domain (for linear maps). So $T_pM = \ker(dg_p)$.

Example. The sphere again. The sphere $S^n \subset \mathbb{R}^{n+1}$ is defined by setting $g = 1$ where $g(x) = \langle x, x \rangle$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on the sphere. Differentiating $g$ at a point $x$ of the sphere yields $dg(\omega)(v) = 2\langle \omega, v \rangle$. 1 is a regular value. So the tangent space at $\omega$ is given by the set of vectors $v$ perpendicular to $\omega$.

The tangent bundle, embedded version.

For each $p \in M$, form the tangent space $T_pM$. As $p$ varies, we get a family of distinct vector spaces which varies smoothly with $p$ (the precise meaning of this phrase to be clarified later). The tangent bundle is their union:

$$TM = \bigcup_{p \in M} T_pM$$

It can be turned into an embedded submanifold of $\mathbb{E} \times \mathbb{E}$ by placing the $T_pM$'s in the second $\mathbb{E}$, and the points of $M$ in the first $\mathbb{E}$, i.e. by identifying $T_pM$ with $\{ p \} \times T_pM$.

Example. The sphere. $TS^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ consists of those pairs $(x, v)$ such that $|x|^2 = 1$ and $\langle x, v \rangle = 0$. It is an embedded submanifold of dimension $2n$, and we have just defined it also as a level set for a map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R} \times \mathbb{R}$.

2. The tangent space to an abstract manifold

We now define the tangent space to an abstract manifold.

To see why we want to do this, it helps to recall a few of the abstract manifolds: the projective spaces, Grassmannians, and flat torii.

We will follow the same definition as before. The tangent space will be made up of derivatives of curves passing through $m$. But because the curves lie in some abstract topological space, we cannot treat them like vectors in some $\mathbb{R}^N$. Instead we build the vector space that the tangents to the curves live in out of the curves themselves.

Let $p \in M$, $M$ a smooth manifold, and let $\phi : M \to \mathbb{R}^n$ be a coordinate chart (smooth relative to the smooth structure on $M$!) and covering $p$. Let $c_1, c_2 : (-\epsilon, \epsilon) \to M$ be two smooth curves passing through $p$ at time $t = 0$.

Definition 2.1. (PROVISIONAL) $c_1 \sim_\phi c_2$ iff $\phi \circ c_1$ and $\phi \circ c_2$ have the same tangent vector at $t = 0$ in $\mathbb{R}^n$. We write $c'_1(0)$ for the equivalence class which contains $c_1$ and $c_2$, and call it a tangent vector at $p$. 

The affine tangent space to the graph, viewed as a 1-manifold, is the tangent line of calculus. If $(x_0, y_0)$ lies on the graph, then it passes through the graph.
This definition depended on a chart. For it to be a good definition, what we call “intrinsic” it must be independent of the choice of chart.

**Lemma 2.2.** Let $M$ be a smooth manifold, $p \in M$ and $\phi$ a chart whose domain covers $p$. Then the equivalence relation $\sim_\phi$ on curves passing through $p$ does not depend on the choice of chart: $c_1 \sim_\phi c_2 \iff c_1 \sim_\psi c_2$, for $\psi$ another chart whose domain covers $p$.

**Proof.** Use, as always, the transition maps $\psi^{-1} \circ \phi$.

**Definition 2.3.** A tangent vector at $p$ is an equivalence class for a smooth curve $c$ passing through $p$. We write: $v = c'(0)$, or $v = [c]$ or $v = [c]_p$ for the resulting equivalence class of curves. The tangent space $T_p M$ at $p$ is the set of these equivalence classes $c'(0)$.

**Proposition 2.4.** The tangent space at $p$ has a natural linear structure.

**Proof.** Let $v, w \in T_p M$. To add them, choose any chart. Let $c_1, c_2$ be the curves which represent them, and write $V = d\phi(v), W = d\phi(w)$ to mean the derivative of $\phi \circ c_1, \phi \circ c_2$ at $t = 0$. Choose any smooth curve in $\mathbb{R}^n$ tangent to $V + W$, for example $\gamma(t) = x + t(V + W)$ where $x = \phi(p)$. We represent $v + w$ as the equivalence class $\phi^{-1} \circ \gamma'(0)$. To see that this definition of vector addition is independent of chart, note that if $\psi$ were another chart, then the derivatives of $\psi \circ c_1, \psi \circ c_2$ at $t = 0$ would be given by $L(V) + L(W) = L(V + W)$ where $L = d(\psi \circ \phi^{-1})$ evaluated at the point corresponding to $p$.

For scalar multiplication, note that if $v$ is represented by $c(t)$ in $\mathbb{R}^n$, then $c(\lambda t)$ will represent $\lambda v$. Now this carries over to the manifold, and is chart-independent.

**Proposition 2.5.** If $E$ is a f.d. real vector space, and $p \in E$ then we have a canonical isomorphism $T_p E \cong E$.

**Proof.** A curve $c(t)$ through $p$ is equivalent to a parameterized line $p + tv$ where $v = c'(0)$. This correspondence sets up a bijection between the equivalence classes of curves and the elements of $v$. QED

**Terminology** This important isomorphism is called ‘translation’ or “parallelism” or “parallel translation”. Algebraically it is implemented by adding $p$ to $v \in E$.

**Differential** If $F : M \to X$ is a smooth map between manifolds we define its differential at $p$.

**Definition 2.6.** The differential of $F$ at $p$, variously denoted as $dF(p)$ or $F_p$ is the map $T_p M \to T_{F(p)} M$ which takes the vector $v$ described by the curve $c$ to the vector $dF(p)(v)$ described by the curve $F \circ c$.

**Proposition 2.7.** The differential $dF(p)$ is a linear map $T_p M \to T_{F(p)} M$.

**Proof.** Work using charts, $\phi, \psi$. In the coordinate picture $F$ becomes the (nonlinear) map $\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^k$. Its differential, as defined via curves, is the usual differential which is linear.

**Important example.** If $\phi = (x^1, \ldots, x^n) : M \to \mathbb{R}^n$ is a chart, then its differential $d\phi_p : T_m M \to T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n$ is a linear isomorphism for each $p$ in the
Moreover, if \( \phi \) is a chart and \( \partial \phi \) is the coordinate basis \( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \), then the inverse image of the standard basis \( e_1, e_2, \ldots, e_n \) for \( \mathbb{R}^n \) under this isomorphism is written \( \frac{\partial}{\partial x^i} \). The space \( \phi^{-1}(U) \) is referred to as the coordinate basis for \( T_xM \).

**Example.** Let \( M \subset \mathbb{E} \) be an embedded submanifold. Show that the inclusion \( i : M \rightarrow \mathbb{E} \) is smooth. Discuss and characterize the differential of the inclusion \( di_m: T_mM \rightarrow T_{i(m)}\mathbb{E} = \mathbb{E} \).

**EXER.** Show that upon restricting the projection map \( (x, y, z) \mapsto (x, y) \) to the sphere \( S^2 \) we get a smooth map \( S^2 \rightarrow \mathbb{R}^2 \). Find those points of the sphere where the differential of this map is NOT a linear isomorphism. (Such points are called “critical points”.)

The tangent bundle \( TM \) inherits a manifold structure from the smooth structure of \( M \). Suppose \( \phi: U \subset M \rightarrow V \subset \mathbb{R}^n \) is a chart. Set \( T_U M = \bigcup_{p \in U} T_p M \subset TM \). Use \( \phi \) and prop ?? to identify points of \( T_U M \) with \( V \times \mathbb{R}^n \) by sending \( (q, v) \) to \( \phi(q), d\phi_q(v) \).

If \( \psi \) is another chart, then on the overlap we have \( (x, V) \mapsto (\phi \circ \psi^{-1}, d(\phi \circ \psi^{-1})(q)(V)) \) which is a smooth map, since \( \phi \circ \psi^{-1} \) is smooth and its Jacobian \( d(\phi \circ \psi^{-1})(q) \) is a smoothly varying matrix. QED

**Notation.** We will often write points of the tangent bundle as pairs \((x, v)\) with \( x \in M \) and \( v \in T_xM \).

### 3. Vector Bundles

The tangent bundle, together with its canonical projection \( TM \rightarrow M \) sending \((x, v) \rightarrow x\), i.e. sending a tangent vector to the point at which it is attached, is the premier example of a smooth vector bundle.

**Warning.** The map \((x, v) \rightarrow v\) does not make sense for or general \( M, v\) – or the space it lives in \( T_xM \) – really depends on \( x \): there is not one single \( n\)-dimensional vector space \( V \) in which the different \( v\)’s lie!

Heuristic def of a v.b. (vector bundle). A v.b. over a topological space \( M \) is a smooth assignment \( m \mapsto E_m \) of vector spaces to points of \( M \).

**Full definition.** A real rank \( n \) vector bundle over a topological space \( M \) is another topological space \( E \) together with a continuous surjection \( \pi: E \rightarrow M \) enjoying the following properties. \( M \) is covered by open sets \( U_\alpha \) such that over \( U_\alpha \) we have homeomorphisms: \( \phi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^n \), called local trivialisations satisfying \( \pi_1(\phi_\alpha(\xi)) = \pi(\xi) \) where \( \pi_1: U_\alpha \times \mathbb{R}^n \rightarrow U_\alpha \) projects onto the first factor, i.e.: such that:

\[
\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{R}^n \\
\downarrow & & \downarrow \pi_1 \\
U_\alpha & \xrightarrow{\pi} & U_\alpha
\end{array}
\]

Moreover, if \( U_\alpha \cap U_\beta \neq \emptyset \) then the homeomorphism \( \phi_\alpha \circ \phi_\beta^{-1}: (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{E} \), which necessarily is of the form \((x, v) \mapsto (x, g(x, v))\) has the form \( \phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)(v)) \) where \( g_{\alpha\beta}: (U_\alpha \cap U_\beta) \rightarrow GL(n) \) is a continuous map into the space \( GL(n) \) of invertible linear maps from \( \mathbb{R}^n \rightarrow \mathbb{R}^n \).

**Terminology** The \( g_{\alpha\beta} \) are called the clutching functions or transition maps for the vector bundle.
More terminology. The fibers of $E$ are the sets $\pi^{-1}(m) := E_m$. Because the $g_{\alpha \beta}(x)$ are linear functions, each fiber has the structure of a vector space, independent of choice of local trivialization. So it makes sense to scalar multiply an element of $E$, or to add to elements which lie in the same fiber. We cannot consistently add elements lying in different fibers. Each $E_m$ has its zero vector $O_m$ and the map $m \mapsto O_m$ is called the zero section. It is a continuous map $s : M \to E$ satisfying $\pi \circ s = s$. More generally, any map $M \to E$ satisfying $\pi \circ s$ is called a section of $E$.

Definition 3.1. A smooth vector bundle $\pi : E \to M$ is a vector bundle over a smooth manifold, which is a smooth manifold in its own right, and is such that the local trivializations and clutching functions are all smooth.

Examples Big example. The tangent bundle of a manifold.

Examples. If $M \subset \mathbb{R}^N$ is an embedded submanifold, then we can define its normal bundle $\nu M$ by setting $\nu_m(M) = (T_m M)^\perp$ where we use the standard inner product to take perpindiculars.

Examples. Tensor product of vector bundles. Direct sum of vector bundles. Duals to vector bundles. If $E, F \to M$ are (smooth) vector bundles, then so are $E \otimes F, E \oplus F, E^*$ and $Hom(E, F)$. Their fibers are $E_m \otimes F_m, E_m \oplus F_m, E_m^*$ and $Hom(E_m, F_m)$. Their clutching functions are $g_{\alpha \beta} \otimes h_{\alpha \beta}, g_{\alpha \beta} \oplus h_{\alpha \beta}, g_{\alpha \beta}^*, \ldots, \ldots$.

The Trivial bundle Is the bundle $M \times \mathbb{R} \to M$.

Exercise. For $M \subset \mathbb{R}^N$ an embedded submanifold, prove that $\nu(M) \oplus TM = M \times \mathbb{R}^N$ is the trivial bundle.

Vector bundle maps. A v.b. map $G : E \to F$ between two vector bundles over the same space $M$ is a continous map which is linear on each fiber. If $E \to M$, $F \to N$ and $g : M \to N$ then a vector bundle map over $f$ is a map $E \to F$ such that $G(E_m) \subset F_{g(m)}$ and $G$ is linear on each fiber.

Similarly, we define smooth vector bundle maps if $E, F$ are smooth vector bundles.

Proposition 3.2. If $g : M \to N$ is smooth, then $dg : TM \to TN$ is a smooth vector bundle map covering $g$. 