The adjoint actions.

Lecture loosely based on notes of Lawson.

Intrinsic to $(\mathfrak{g}, G)$ are a collection of three maps, intimately intertwined and basic to the whole theory. They are the various adjoint actions:

$$ad : \mathfrak{g} \to der(\mathfrak{g})$$

$$Ad : G \to Aut(\mathfrak{g})$$

and

$$AD : G \to Aut(G)$$

We generally write the value of any one of these using a subscript, eg. Ad_g not $Ad(g)$. The three 'ads" fit conveniently into a big commutative diagram:

$$\begin{array}{ccccccc} 0 & \to & \mathfrak{z} & \to & \mathfrak{g} & \to^{ad} & der(\mathfrak{g}) \\ & & \downarrow & & \downarrow \exp & & \downarrow \exp \\ 1 & \to & Z_0 & \to & G & \to^{Ad} & Aut(\mathfrak{g}) \\ & & \uparrow & & \downarrow Id. & & \downarrow ? \\ 1 & \to & Z & \to & G & \to^{AD} & Aut(G) \end{array}$$

In this note we explain these maps and the arrows and spaces of the diagram.

$AD(g) := AD_g$ is conjugation by g: $AD_g(h) = ghg^{-1}$. Alternatively:

$$AD_g = L_g \circ R_{g^{-1}} : G \to G.$$

Each AD_g is Lie group automorphism of G.

We have $AD_g(1) = 1$, so that if we differentiate $AD_g(h)$ with respect to h at $h = 1$ we get a map:

$$Ad_g := d(AD_g)_1 : T_1G = \mathfrak{g} \to T_1G = \mathfrak{g}.$$

Each Ad_g is Lie algebra automorphism of $\mathfrak{g}$. (See below for the definition of Lie algebra automorphism.) Put together, they form the adjoint representation of G which is a linear representation of G on $\mathfrak{g}$, i.e. a Lie group homomorphism $G \to Gl(\mathfrak{g})$. From the definition of AD we have the alternative expression

$$Ad_g = (dL_g) \circ dR_{g^{-1}} : \mathfrak{g} \to \mathfrak{g},$$

where the derivatives are evaluated at 1 and g^{-1}.

Finally ad is the bracket operation:

$$ad_X(Y) = [X, Y], X, Y \in \mathfrak{g}.$$

At first glance, the most mysterious of the three is the middle one, Ad.

Matrix case of Ad. You already know Ad in the matrix case: $G = Gl(n)$ or G a closed subgroup of $GL(n)$. For as we have seen before, in that case L_g, R_g are the restrictions of linear maps on $gl(n)$. It follows that in the matrix case

$$Ad_g(X) = gXg^{-1},$$

so has the same form as AD_g, just a different domain.

Important is the commutativity of the diagram:

$$exp(ad_X) = Ad_{exp(X)} \tag{1}$$

$$exp(Ad_g X) = AD_g(exp(X))$$

In the matrix case these facts are easily seen algebraically. For example, the second fact is easily checked by:

$$\begin{aligned} exp(Ad_g(X)) &= I + (gXg^{-1} + (1/2)(gXg^{-1})^2 + \ldots \\ &= g(I + X + (1/2)X^2 + \ldots)g^{-1} \\ &= AD_g(exp(X) \end{aligned}$$

To check commutativity in the general case it helps to better understand the last column of spaces in the diagram. $Aut(G)$ is the group of Lie automorphisms of G. It is a Lie group in its own right. $Aut(\mathfrak{g})$ is the group of LIe algebra automorphisms of $\mathfrak{g}$. A Lie algebra automorphism of $\mathfrak{g}$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ which preserves brackets: $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$. Thus, for each $g \in G$ we have that Ad_g is a Lie algebra automorphism of $\mathfrak{g}$. Finally $der(\mathfrak{g})$ is the vector space of all derivations of $\mathfrak{g}$. A *derivation* of $\mathfrak{g}$ is by definition a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta([x,y]) = [\delta(x), y] + [y, \delta(x)].$$

Note that $Aut(\mathfrak{g}) \subset GL(\mathfrak{g})$ and $der(\mathfrak{g}) \subset gl(\mathfrak{g})$. $Aut(\mathfrak{g})$ is a matrix Lie group and its Lie subalgebra is $der(\mathfrak{g})$. Thus the matrix exponential defines the map

$$exp : der(\mathfrak{g}) \to Aut(\mathfrak{g})$$

.

The images of the various *ad*'s, i.e $ad(\mathfrak{g})$, $Ad(G)$ and $AD(G)$ are called "inner": inner derivations and inner automorphsims of $\mathfrak{g}$ and of G.

Why is the bottom right arrow a question mark in the diagram? The reason is that in general not every Lie algebra homomorphism exponentiates up to define a Lie group homomorphism, so that there is no "*exp*": $Aut(\mathfrak{g}) \to Aut(G)$. The classic example is the circle. For $G = S^1$ we can identify $\mathfrak{g} = \mathbb{R}$ so that $exp(t) = e^{it}$. Any linear map $\mathbb{R} \to \mathbb{R}$ is a Lie algebra automorphism of $\mathbb{R}$. Any linear map has the form $t \mapsto ct$. Only those maps for which c, is an integer induce Lie group homomorphisms of the circle. And only the trivial case $c = 1$ induces Lie algebra automorphism of the circle.

However, if G is simply connected (like $\mathbb{R}$) then it is true that every Lie algebra automorphism of $\mathfrak{g}$, or more generally, every Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ exponentiates to define a Lie group homomorphism $G \to H$. (H does not have to be simply connected.)

The kernels.

The kernel of AD, traditionally written Z, is the center of G: the subgroup made up of those the elements which commute with any other element of G. The kernel of ad is called the center of $\mathfrak{g}$: the subLie algebra made up of those elements of $\mathfrak{g}$ which bracket-commute with every other element of $\mathfrak{g}$. Finally, the kernel of Ad, traditionally written $\mathfrak{z}$, consists of those elements of G which act trivially in the adjoint representation.

We have $Lie(Z) = \mathfrak{z}$.

Case of $G = SU(2)$. Then $\mathfrak{g} = \mathbb{R}^3$ as a linear space and $Ad(G) = SO(3)$. The kernel of Ad is $\pm I \subset G$ and agrees with the center of G. These are finite groups, which corresponds to the fact that $\mathfrak{z} = 0$ in this case.