Universal Covers. Functoriality: $\mathfrak{g} \to G$

The collection of all Lie algebras forms a category whose arrows are Lie algebra homomorphisms. The collection of all Lie groups forms another category whose arrows are Lie group homomorphisms. Linearization (or differentiation) defines a functor:

$$Lie : LieGroups \to LieAlgebras.$$

We have described this functor on the level of objects: how it associates to each Lie group a Lie algebra. We need how to associate a Lie algebra homomorphism to a Lie group homomorphism. If $\phi : G \to H$ is a Lie group homomorphism then its linearization defines a Lie algebra homomorphism, say

$$\phi' = Lie(\phi) : \mathfrak{g} \to \mathfrak{h}.$$

If we identify $\mathfrak{g}, \mathfrak{h}$ with the tangent spaces at the identity, then $\phi' = d\phi(1) : T_1 G \to T_1 H$, also written $\phi_{*,1}$ or simply ϕ_1, depending on the text is a linear map. This definition makes the linearity of ϕ' clear. To get that ϕ' is a Lie algebra homomorphism, use the definition of the Lie algebra as left-invariant vector fields. There is a minor problem here: the image $\phi_* \xi$ of a left-invariant vector field ξ on G will not be a vector field on all of H if ϕ is not onto. But the problem is easily surmounted: $\phi_* \xi$ extends uniquely to a left-invariant vector field on all of H.

In this way we get a functor, really just differentiation, from Lie groups to Lie algebras. We would like to say that we can invert this functor by integration.

However this will not be true! There are two related reasons.

1. There are typically many Lie groups sharing the same isomorphic Lie algebra. For example, $\mathbb{R}$ and S^1 have the commutative Lie algebra $\mathbb{R}$ as their Lie algebra. And $SU(2)$ and $SO(3)$, and $O(3)$ all have $\mathbb{R}^3, \times$ (cross-product) as their LIe algebra. Given a $\mathfrak{g}$, which one of these possible many G's with $Lie(G) = \mathfrak{g}$ do we assign $Int(\mathfrak{g}) = Lie^{-1}(\mathfrak{g})$?

2. There are Lie algebra homomorphisms ϕ' which do not integrate up to give Lie group homomorphisms. For example: take the circle $G = S^1$, thought of as unit modulus complex numbers. Identify its Lie algebra $\mathfrak{g}$ with $i\mathbb{R}$. The exponential map sends $it \in i\mathbb{R}$ to e^{it}. Any linear map $i\mathbb{R} \to i\mathbb{R}$ is a Lie algebra automorphism of $\mathbb{R}$. Any of these linear maps has the form $it \mapsto ict$, c real. Only those maps for which c, is an integer induce Lie group homomorphisms of the circle.

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Both difficulties vanish upon passing to the universal cover. But then, we better understand the universal cover.

Theorem 1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the universal cover $\tilde{G}$ of G is itself a Lie group in such a way that the covering map $\tilde{G} \to G$ is a Lie group homomorphism. $\tilde{G}$ is the unique simply connected Lie group having $\mathfrak{g}$ as its Lie algebra. And if $\phi' : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism where $\mathfrak{h}$ is the Lie algebra of a connected Lie group H then there is a unique Lie group homomorphism $\tilde{G} \to H$ with $Lie(\phi) = \phi'$.*

To understand the theorem, we need to understand the universal cover $\tilde{M}$ of a path-connected topological space M. I will go through this quickly.

References. Greenberg, esp . 1st edition. (Greenberg taught here when I first came here. He retired in around 1994.) Vassiliev.

Some homotopy. Def of homotopy between maps. Between curves. Fact that if two maps from a compact into a manifold are sufficiently C^0 close then they are homotopic. Idea of proof.

Based path space of a connected manifold M. Notation $P(m, m_0)$. Equivalence relation $\sim$ w both ends fixed. Quotient $\tilde{M} = P(M, m_0)/\sim$ as model for universal cover. Concatenating paths. Fundamental groupoid of a space: neither point fixed.

Properties of $\tilde{M}$.

Covering property. A continuous onto map $\pi : Y \to X$ between topological spaces is called a 'covering map' if there is a discrete space F and a covering of X by open sets $\{U_a\}$ such that for each open set U_a there is a homeomorphism $\pi^{-1}(U_a) = U_a \times F$. We show that $\tilde{M}$ has the covering property. The topology on $P(M, m_0)$ is the uniform topology on curves. (Put any metric on M compatible with its topology.) The topology on $\tilde{M}$ is then the quotient topology for $\sim$ on M. If γ is a path and if γ_1 lies in a uniform nbhd of the path, then, as mentioned earlier, γ_1 is homotopic to γ. It follows that the fibers $\pi^{-1}(m)$ are discrete. If m_1, m_2 are two different points, fix a path γ_{12} joining m_1 to m_2. Then concatenating this path to any path from m_0 to m_1 defines an invertible map $\pi^{-1}(m_1) \to \pi^{-1}(m_2)$.

PICTURE HERE.

Take $F = \pi^{-1}(m_0)$. Concatenation of elements of F by a fixed path γ_m connecting m_0 to m then defines a homeomorphism from F to $\pi^{-1}(m)$.

Now fix $m \in M$ and take for U a "simply connected nbhd of m and connect each $y \in U$ to m by a fixed path c_{my} lying in U. (Think of U as a disc with center m and take radial paths.) The map $U \times F \to \pi^{-1}(U)$ defined by $(y, [\gamma]) \mapsto [c_{my} * \gamma_m * \gamma]$ is a homeomorphism, and shows that $\tilde{M}$ is a covering space.

Smoothness: Since M is a smooth manifold and F is discrete, we can take the local trivializations just defined to be a diffeomorphisms. , thus defining a smooth structure on $\tilde{M}$.

Universal Mapping. Covering Homotopy property. Review this ... Let $F_0, F_1 : C \to M$ be homotopic maps and let $\tilde{F}_0 : C \to \tilde{M}$.... Universal mapping property of universal cover.

Relation to monodromy and DEs.

Relation to fundamental group.

Case in which M is a connected Lie group G.

Get $\pi_1(G) =$ Abelian, from 2nd HW exer: $\pi_1(G)$ realized as discrete and normal.

construction. Two curves γ_0, γ_1 are called *homotopic* if there is a curve of curves joining them. More precisely: suppose $\gamma_0, \gamma_1 : I \to M$ are continuous curves. A homotopy between γ_0 and γ_1 is a map $f : I \times I \to M$ with the property that $f(0, t) = \gamma_0(t)$ and $f_1(t) = \gamma_1(t)$. If we fix the initial point, $m_0 = \gamma_0(0) = \gamma_1(0)$ then we will insist that the $f(I \times 0) = m_0$.

Notation: $f(s, t) = \gamma_s(t)$.

Fix m_0 as above. Write $\tilde{M}$ for the space of all homotopy classes of curves in M starting at M. Define $\pi : \tilde{M} \to M$ by $\pi([\gamma] = \gamma(1)$. Since M is a manifold it is covered by simply connected open nbhds. If U is such a nbhd, and $p \in \tilde{M}$ is represented by $[\gamma]$. Then I claim that $\pi^{-1}(U)$ is a disjoint union of sets each of which is a copy of U.

Simply connectedness of $\tilde{M}$.

Defn of cover.

Of universal cover.

Def of $\pi_1(M, m_0)$.

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The exponential is a local inverse in a sense that: **Exercise:** Prove that for $X \in \mathfrak{g}$ and $\phi : G \to H$ as above the equality

$$exp(\phi'(X)) = \phi(exp(X)$$

holds.