Loose ends.

Bracket on $gl(n)$.

Recall that for $gl(n) =$ all $n \times n$ matrices and is the tangent space to $G = GL(n)$ at any point A.

Left multiplication: $G \to G$ is $L_A(B) = AB$ and is **linear** in B if we allow B for vary over $gl(n) \supset G$. Consequently, for $X \in gl(n) = T_1 G$ we have

$$dL_A(X) = AX$$

and

$$X^L(A) = AX.$$

From an exercise in the manifolds class then:

$$[X^L, Y^L] = (XY - YX)^L$$

showing that the Lie bracket of left-invariant vector fields on G corresponds to the operator bracket on $gl(n)$.

I will do the exercise here, now. The X^L, Y^L are well-defined as vector fields on all of the vector space $E = gl(n)$. We have the general formula

$$[V, W](p) = DW_p \cdot V(p) - DV_p \cdot W(p)$$

$$\begin{aligned} [X^L, Y^L](A) &= D(Y^L)_p \cdot X^L(A) - D(X^L)_p \cdot Y^L(A) \\ &= X^L(A)Y - Y^L(A)X \\ &= AXY - AYX \\ &= A(XY - YX) \\ &= (XY - YX)^L(A) \end{aligned}$$

for the Lie bracket of two vector fields $V, W : E \to E$ on a vector space. Now our vector fields are linear. So that $D(X^L)_A \cdot h = hX$ and similarly for the differential of Y^L. It follows that

(1) $$a \quad b$$

(2)