the exponential map.

Recall that we can identify $\mathfrak{g}$ with the space of one-parameter subgroups. It now helps to be explicit, and commit ourselves to one model of $\mathfrak{g}$. We take it to be T_1G. Then for $X \in T_1G = \mathfrak{g}$ we write X^L for the corresponding left-invariant vector field and $exp(tX)$ for the corresponding one-parameter subgroup. Thus for $t \in \mathbb{R}$ we have that $exp(tX) \in G$. Recall that $exp(tX) = \phi(t)$ is the unique solution to the ODE $d\phi/dt = X^L(\phi(t)$ with initial condition $\phi(0) = 1 \in G$.

For $X \in \mathfrak{g}$ we have $exp(X) = \phi(1)$. and a well-defined map

$$exp : \mathfrak{g} \to G.$$

PICTURE HERE.

Examples

Eg. 1. $G = \mathbb{R}^n$. $\mathfrak{g} = \mathbb{R}^n$. $exp(tv) = tv$.

Eg. 2 $G = S^1$, $\mathfrak{g} = i\mathbb{R}$. $exp(i\theta) = \cos(\theta) + i\sin(\theta)$.

Eg. 3. $G = S^3 = Sp(1)$. $\mathfrak{g} = im(\mathbb{H})$. Any purely imaginary quaternion n of unit length satisfies $n^2 = -1$. The same argument as for the exponential in example 2, the circle, then yields $exp(\theta n) = \cos(\theta)1 + n\sin(\theta)$ for θ real. Now any purely imaginary quaternion h can be written $h = \theta n$ with $n = h/|h|$ a unit pure imaginary quaternion.

Eg. 4. $G = Gl(n)$. $\mathfrak{g} = \mathfrak{g}l(n)$. $exp(A) =$ usual matrix exponential.

Proposition 1. *The exponential is a local diffeo of a nbhd of o in $\mathfrak{g}$ onto a nbhd of the identity in G*

Proof. IFT.

As a cor. to this prop. we have two types of standard coordinates on any Lie group. First, fix coord x^i on the lie algebra by fixing a basis E_i for $\mathfrak{g}$.

1) exponential coordinates of the 1st kind: $(x^1, \dots, x^n) \mapsto exp(\Sigma x^i E_i)$

2) exponential coordinates of the 2nd kind: $(x^1, \dots, x^n) \mapsto exp(x^n E_n) \dots exp(x^1 E_1)$

We can do most of the Lie group operations on the Lie algebra. For example

$$exp(X)^{-1} = exp(-X).$$

so that inversion is $X \mapsto -X$ in the algebra.

Written out in either type of coord system multiplication is an analytic map. Here is a coord free way to say this:

Proposition 2. *Define $\mu : \mathfrak{g} \times \mathfrak{g} - - \to \mathfrak{g}$ by $exp(X)exp(Y) = exp\mu(X,Y)$. Then μ is an analytic map near $(0,0)$. Its Taylor expansion up to 2nd order is: $\mu(X,Y) = X + Y + \frac{1}{2}[X,Y]$*

.

Proof. I will just prove that μ is smooth. See Duistermaat or Varadarajan for proofs of analyticity and the full Taylor expansion.

We have the diagram:

$$\begin{array}{ccc} \mathfrak{g} \times \mathfrak{g} & ? & \mathfrak{g} \\ \downarrow & & \downarrow \\ G \times G & \to & G \end{array}$$

where the bottom arrow is multiplication and the arrows going down are the exponential map and its pairwise product $(\exp, \exp) : \mathfrak{g} \times \mathfrak{g} \to G$. The question mark is the desired map μ. The two downward arrows are local diffeos, so have local inverses which we call "log and $(\log, \log)$. From the diagram $\mu = \log \circ m \circ (\log, \log)$

is the composition of smooth maps. where we have written log for the local inverse of exp.

****

break ?

students present HWs

****

The Lie algebra generates the group.

Theorem 1. *Let U be any nbhd of 0 in $\mathfrak{g}$. Then $exp(U)$ generates the identity component of G.*

Some explanation is in order. Recall that a subset $S \subset \Gamma$ of a general group Γ generates Γ if any element of Γ is expressible as a (finite!) product of elements of S.

Definition 1. *A topological group is a group which is also a topological space and is such that the multiplication map $G \times G \to G$ and the inversion map $iG \to G$ are continuous maps.*

Thus Lie groups are topological groups but not all topologicla groups are Lie groups. The real nature of the 1st HW problem is

HW: find a topological group which is not a Lie group; i.e. it admits no smooth structure.

Definition 2. *The identity component G^0 of a topological group is the component of G containing the identity.*

Proposition 3. *. Let G be a topological group. Then G^0 is a topological subgroup of G which is also normal. And if $U \subset G^0$ is any nbhd of the identity then U generates G^0.*

Pf. I will just do the case of a manifold in which case connected is the same as path connected. Suppose that g_1, g_2 are in the path connected component of the identity. L et $\gamma_1, \gamma_2 : [0,1] \to G$ be the corresponding paths connecting the identity to these points. The product path $\gamma_3(t) = \gamma_1(t)\gamma_2(t)$ connects 1 to $g_1 g_2 = \gamma_1(1), \gamma_2(1)$. This shows that G_0 is closed under multiplication. Using $\gamma^*(t) = \gamma(t)^{-1}$ shows that it is closed under inversion.

I will leave it to you to show that G^0 is normal.

For the last part, consider the subgroup generated by U. Since U is a nbhd this subgroup is open. It is also closed. So it must equal G^0.