Lie groups to Lie algebras.

Lie used the word the 'infinitesimal group' of one of his groups for what we now call, thanks to E. Cartan, its 'Lie algebra'.

There are (at least) four useful equivalent definitions of the LIe algebra $\mathfrak{g}$ of a Lie group G.

1. **tangent vectors.** $\mathfrak{g} = T_e G$, the tangent space to G at its identity 1.

2. **invariant vector fields.** $\mathfrak{g} =$ space of left-invariant vector fields on G.

3. **one parameter subgroups.** Def. A one-parameter subgroup of G is a Lie homomorphism $\mathbb{R} \to G$. Then $\mathfrak{g} =$ space of one-parameter subgroups in G.

4. **invariant flows.** $\mathfrak{g} =$ space of flows on G which commute with left translation: $\Phi^t(gh) = g\Phi^t(h)$.

For def. 4: , recall that a 'flow' on a manifold is a smooth $\mathbb{R}$ action. That is to say, it is a map $\Phi : \mathbb{R} \times M \to M$ such that $\Phi(t+s, m) = \Phi(t, \Phi(s, m))$. We also write $\Phi^t(m) = \Phi(t, m)$. then $\Phi^t : M \to M$ and $\Phi^{t+s} = \Phi^t \circ \Phi^s$. So that flows define homomorphisms from $\mathbb{R}$ to the group $Diff(M)$.

I will now go through the advantages and disadvantages of these definitions, and their equivalences.

1. The tangent space of any manifold at a point is a vector space whose dimension is that of the manifold. Thus 1. immediately gives $\mathfrak{g}$ the structure of a finite-dimensional vector space, of dimension $dim(G)$. But it does not give the Lie bracket. And it only gives an extremely limited relation between $\mathfrak{g}$ and G.

2. The space of ALL vector fields $\Gamma(TM)$ on a manifold M is an infinite dimensional Lie algebra under the standard Lie bracket of vector fields. With def. 2 we have $\mathfrak{g} \subset \Gamma(TG)$ as a Lie subalgebra. So we get the Lie bracket structure immediately. But why is it finite-dimensional? In 2. the global connection between G and $\mathfrak{g}$ becomes evident.

3. In this definition we apparently lose the vector space structure and the Lie bracket structure. But we establish intimate relations with the group structure.

4. Similar complaints as for 3. This def. is not as much used as the other three , but forms the essential bridge between 2 and 3.

REVIEW. Of $T_m M$. Of $\Gamma(TM)$. Of $[X, Y] = [\Sigma X_a \frac{\partial}{\partial x^a}, Y_b \frac{\partial}{\partial x^b}]$ formula.

$1 \implies 2$. Left translation establishes a canonical linear isomorphism between different tangent spaces of G. Thus: $(dL_g)_1 : T_1 G \to T_g G$. For $X \in T_1 G$ define $X^L(g) = dL_g X$. Then X^L is left-invariant:

$$\begin{aligned} L_{g*} X^L(h) &= dL_g X^L(L_g^{-1} h) \\ &= dL_g X^L(g^{-1} h) \\ &= dL_g (dL_{g^{-1}h}) X \\ &= dL_g dL_{g^{-1}} dL_h X \\ &= dL_h X \\ &= X^L(h) \end{aligned}$$

$2 \implies 1$. If ξ is a left-invariant vector field set $X = \xi(1) := ev_1(\xi)$. Check that $\xi = X^L$.

Note that $X \mapsto X^L$ and $\xi \mapsto ev_1(\xi)$ are both linear and are each others inverses.

IN MATRIX TERMS. Suppose $G = Gl(n, \mathbb{R})$. Being an open set of $\mathfrak{g} = \mathfrak{g}l(n) = hom(\mathbb{R}^n, \mathbb{R}^n)$ we have that $T_1 G = gl(n)$. Now $(g, A) \to gA$ is linear in A, so that $dL_g(X) = gX$. Thus, for $X \in gl(n)$ a matrix we have $X^L(g) = gX$ the corresponding left-invariant vector field.

A similar trick works for Lie subgroups of $Gl(n)$. Example. $G = SO(3)$. Compute: $T_I G = \{\xi : \xi^T + \xi = 0\}$. This is 3-d. Now a skew-symmetric matrix $X \in so(3)$ defines a left-invariant vector field X^L on $SO(3)$ by $X^L(g) = gX$.

For 2 $\implies$ 4: we first

REVIEW: the notion of the flow of a vector field.

$X \in \Gamma(TM)$. Then its flow, written $\Phi^t_X : M \to M$ is a one-parameter group of diffeos of M. It is defined by $\Phi^t_X(m) = \gamma(t)$ where $\gamma(t)$ is the unique solution to the ODE $d\gamma/dt = X(\gamma(t))$, and $\gamma(0) = m$. We have $\Phi^{t+s}_X = \Phi^t_X \circ \Phi^s_X$.

Notation. We also write $\Phi(t, x)$ for $\Phi^t(x)$.

By definition, $d\Phi_X(t, m)/dt|_{t=0} = X(m)$

EXER. Show that if $\psi : M \to M$ is a diffeo satisfying $\psi_* X = X$ then ψ commutes with the flow of X.

2 $\implies$ 4. Given a left invariant vector field ξ let $\Phi = \Phi^t_\xi$ denote its flow. Because ξ commutes with all the L_g's we have, by the EXER, that $\Phi^t(gh) = g\Phi^t(h)$.

4 $\implies$ 2. For Φ such a flow, set $\xi = d\Phi(t, g)/dt|_{t=0}$.

4 $\implies$ 3. For $\Phi : \mathbb{R} \times G \to G$ such a flow, set $\phi(t) = \Phi(t, 1)$. Because Φ is a flow, we have that ϕ is a one-parameter subgroup.

3 $\implies$ 2. Set $\Phi(t, g) = g\phi(t)$ for ϕ a one-parameter subgroup.

2 $\implies$ 3. Working through the bridge of 4, we see that the one-parameter subgroup corresponding to the left-invariant vector field ξ is the unique solution to:

$$d\gamma/dt = \xi(\gamma(t)); \gamma(0) = I.$$

MATRIX Egs.

For $G = Gl(n)$. Recall that $X^L(\gamma) = \gamma X$ in this G. The ODE defining 3 is then

$$d\gamma/dt = \gamma(t)X$$

with initial condition $\gamma(0) = I$. The solution is

$$\gamma(t) = e^{tX},$$

the matrix exponential.

These are the one-parameter subgroups.

If $G = O(n)$ then $X \in \mathfrak{g}$ is skew-symmetric and one can check by hand that $\phi(t) = exp(tX)$ satisfies $\phi(t)\phi(t)^T = I$, as it must.

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Dictionary.
beginning. more to fill in perhaps , as time goes by ...

Lie group	Lie algebra
G	$\mathfrak{g}$
integrate	differentiate
associative	Jacobi identitiy
Abelian	Abelian: $[X, Y] = 0$
closed subgroup H	subalgebra $\mathfrak{h}$
(Lie) homomorphism (*)	Lie alg. homo.
unipotent	nilpotent
normal subgroup	ideal
quotient group by a closed normal subgroup	guotient Lie algebra by corresponding ideal
product	direct sum
center	$Z = \{z : [z, \mathfrak{g}] = 0\}$
commutator subgroup	derived ideal $[\mathfrak{g}, \mathfrak{g}]$
rep $\rho : G \to Gl(V)$	rep $\rho' : \mathfrak{g} \to gl(V)$
compact	admits Ad-invariant inner prod.
maximal torus	maximum Abel. subalg.

(*): we can in general integrate a Lie algebra homomorphism up to a Lie group homomorphism only for target groups which are simply connected groups.