Extending Lie algebra homomorphisms. The subalgebra : subgroup correspondence.

GIven a Lie group homomorphism $\phi : G \to H$ 'we have a corresponding Lie algebra homomorphism $\phi' : \mathfrak{g} \to \mathfrak{h}$.

Here are three of the four ways to define this ϕ'

1. Using $\mathfrak{g} = T_1 G$. Then $\phi' = d\phi_1 : T_1 G \to T_1 H$. 2. Using $\mathfrak{g}$ = left-invariant vector fields. Then $\phi'(\xi) = \phi_* \xi$. 3. Using one-parameter subgroups $\gamma(t) = exp(tX)$. Then $\phi'(\gamma) = \phi \circ \gamma$.

Now we want to go the other way: given $\phi' : \mathfrak{g} \to \mathfrak{h}$ with G, H connected Lie groups find a $\phi : G \to H$ inducing ϕ'.

If ϕ is to exist one must have that $\phi(exp(tX)) = exp(t\phi'(X))$. This uniquely defines ϕ in a neighborhood of the identity. But it may fail to yield a good definition everywhere.

Example. $\mathfrak{g} = \mathfrak{h} = \mathbb{R}$. $G = H = S^1$. $\phi'(x) = cx$. The exponential $\mathfrak{g} \to G$ is $x \mapsto e^{ix}$ so that this ϕ' would have to induce $\phi(e^{ix} = e^{icx}$ which is not defined as a map from the circle to the circle unless c is an integer.

The solution is to pass to the universal cover.

Theorem 1. *If G is simply connected and H is connected then any Lie algebra homomorphism $\phi' : \mathfrak{g} \to \mathfrak{h}$ induces a unique Lie group homomorphism $\phi : G \to H$.*

A key to proving this theorem is another important theorem

Theorem 2 (subalgebra-subgroup theorem)**.** *To each subalgebra $\mathfrak{K} \subset \mathfrak{g}$ of a connected Lie group G there is a unique injectively immersed connected subgroup $i : K \to G$ with $di_1(T_1 K) = \mathfrak{K}$.*

Warning; K need not be embedded, i.e. it need not be closed. For example Irrational flow on the two torus T^2 defines an immersion of $K = \mathbb{R}$ into $G = T^2$.

Proof of subalgebra-subgroup theorem This theorem is a special case of the Frobenius integrability theorem. The leaf through $1 \in G$ is the immersed subgroup.

Proof of simply connected lift theorem. $G \times H$ forms a Lie group with Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$. The graph of the map ϕ' defines a Lie subalgebra of this direct sum Lie algebra. Consequently, by the subalgebra-subgroup theorem, there is a unique injectively embedded $i : K \to G \times H$ whose image realizes this graph. Consider the composition $i \circ \pi : K \to G \times H \to G$ where $\pi : G \times H \to G$ is the projection. Its differential is 1-1 and onto, being just the compositon $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{K} \to \mathfrak{g}$. Consequently it defines a covering map $K \to G$. But G is simply connected, so this covering map is an isomorphism and we can identify K with G It follows that if $\pi_2 : G \times H \to H$ is the projection then $i \circ \pi_2 : G \to H$ is a Lie group homomorphism and induces ϕ'. This $i \circ \pi_2$ is ϕ. QED.