A minimal route to the classification of simple compact Lie groups.

1. Representation theory generalities. The main tools: averaging, Schur's lemma, characters. A representation is determined by its character. Peter-Weyl theorem statement – used to insure the list of irreps of a G is complete.

2. Representation theory for T a torus.

3. Definitions of the maximal torus, of weights. Roots = weights for the adjoint representation. Fact that a rep is determined by (a) its characters, or (b) its weights. Fact that T is a slice for AD.

4. The Weyl group and its action on $\mathfrak{t}^*$.

5. Representation theory for $SU(2)$.

6. The α-string through β.

7. The Weyl group forms a Coxeter group. Root system axiomatics.

8. Combinatorial classification of root systems.

9. The list.

asides along the way: $G/AD = T/W$, $\mathfrak{g}/Ad = \mathfrak{t}/W$, $W = N(T)/T$.

the classification agrees with the classification of the complex simple Lie algebras: Serre's beautiful short book on that subject.

Structure of Lie groups

We know something about the circle group and $SU(2)$. We indicate how these two groups are the building blocks for understanding everything about a large class of Lie groups. To initiate the discussion, let us see how to put together three circles and three $SU(2)$'s so as to form $U(3)$, the 3×3 unitary group:

$$T = (S^1)^3 = \{ \begin{pmatrix} expi2\pi x_1 & & \\ & expi2\pi x_1 & \\ & & expi2\pi x_1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \}$$

$$SU(2)_{12} = \{ \begin{pmatrix} * & * & \\ * & * & \\ & & 1 \end{pmatrix} \text{ where the } * \text{'s define a matrix in } SU(2) \}$$

$$SU(2)_{13} = \{ \begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix} \text{ where the } * \text{'s define a matrix in } SU(2) \}$$

$$SU(2)_{23} = \{ \begin{pmatrix} *1 & & \\ & * & * \\ & * & * \end{pmatrix} \text{ where the } * \text{'s define a matrix in } SU(2) \}$$

where blank entries means zeros are placed there.

These four subgroups finitely generate all of $U(3)$. Indeed we can dispense any one the $SU(2)$'s , since commutators of the remaining two will generate the excluded one.

This example is a special instance of a general phenomenon for compact connected Lie groups. There is a torus $T \subset G$ inside G called a 'maximal torus'. Associated to T is a distinguished family of $SU(2)'s \subset G$ which together with T generate the entire group. They are called "root $SU(2)$'s. On the Lie algebra level simple bracket relations relating the various $su(2)$'s in $\mathfrak{g}$. Once these bracket relations are understood, the entire structure of the group can be reconstructed.

To illustrate the general phenomenon, it will be better to work with $SU(3) \subset U(3)$. Then the torus is defined by restricting

$$x_1 + x_2 + x_3 = 0 \text{ within } T \subset U(3).$$

The x_i are linear functions on the Lie algebra $\mathfrak{t}$ of the torus. Recall the adjoing action: $Ad_t g = tgt^{-1}$ Writing

$$t = t(x_1, x_2, x_3) = diag(e^{i2\pi x_1}, e^{i2\pi x_2}, e^{i2\pi x_3})$$

for the element indicated in the expression for T we compute (see also Adams) that

$$t \begin{pmatrix} 0 & \xi & 0 \\ -\bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} t^{-1} = \begin{pmatrix} 0 & e^{i2\pi(x_1-x_2)}\xi & 0 \\ e^{-i2\pi(x_1-x_2)}\bar{\xi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that the Lie algebra $su(2)_{12}$ of $SU(2)_{12}$ consists of elements parameterized by ξ as above, plus the intersection of $\mathfrak{t}$ with $su(2)_{12}$.

And:

$$t \begin{pmatrix} 0 & 0 & \xi \\ 0 & 0 & 0 \\ -\bar{\xi} & 0 & 0 \end{pmatrix} t^{-1} = \begin{pmatrix} 0 & 0 & e^{i2\pi(x_1-x_3)}\xi \\ 0 & 0 & 0 \\ e^{-i2\pi(x_1-x_3)}\bar{\xi} & 0 & 0 \end{pmatrix}$$

And:

$$t\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \xi \\ 0 & -\bar{\xi} & 0 \end{pmatrix} t^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i2\pi(x_2-x_3)}\xi \\ 0 & e^{-i2\pi(x_2-x_3)}\bar{\xi} & 0 \end{pmatrix}$$

The three linear functions:

$$\alpha_{12} = x_1 - x_2, \alpha_{13} = x_1 - x_3, \alpha_{23} = x_2 - x_3$$

and their negatives $\alpha_{ji} = -\alpha_{ij}$ which appear in the exponentials in these expressions for Ad_t are called the "roots of $SU(3)$" [relative to our choice of T. They are linear functions on $\mathfrak{t}$ and hence elements of $\mathfrak{t}^*$. The computations above can be summarized by:

$$\xi \in \mathfrak{s}_{ij}, x \in \mathfrak{t} \implies Ad(exp(2\pi x)\xi = \exp(i2\pi\alpha_{ij}(x))\xi$$

Here we have split each $su(2)_{ij}$, $i \neq j$ as

$$su(2)_{ij} = \mathfrak{t}_{ij} \oplus \mathfrak{s}_{ij} = \mathbb{R} \oplus \mathbb{C}.$$

, where $\mathfrak{t}_{ij} = \mathfrak{t} \cap \mathfrak{s}_{ij}$ is diagonal and one dimensional and where $i \neq j$ and $\mathfrak{s}_{ij}$ stands for the subspace of those skew-hermitian matrices whose only nonzero entries are the ij and ji entries. This equation for $Ad(exp(2\pi(x))$, or its differential:

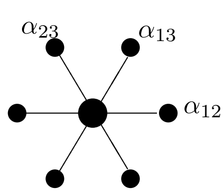
$$[x, \xi] = i\alpha_{ij}(x)\xi, x \in \mathfrak{t}, \xi \in \mathfrak{s}_{ij}$$

are the basic relations defining "roots".

Before we give the general definition of 'root' for a general compact connected Lie group, we continue with $SU(3)$. The roots lie in $\mathfrak{t}^*$ which is two dimensional, so we can draw a pictures. This $\mathfrak{t}^*$ inherits a natural inner product from the Ad-invariant inner product on $su(3)$:

$$\langle X, Y\rangle = tr(XY^*).$$

Here is the picture of $\mathfrak{t}^*$ with its roots:



The one more thing that can be read off of the diagram is that

$$[\mathfrak{s}_{12}, \mathfrak{s}_{23}] \subset \mathfrak{s}_{13}] \text{ plus cyclic permutations}$$

and which corresponds to the fact that the roots satisfy:

$$\alpha_{12} + \alpha_{23} = \alpha_{13}$$

etc.

General Theory

For this section, let G be any compact connected Lie group.

Definition 1. *A maximal torus in G is an embedded Lie subgroup $T \subset G$ which is connected, Abelian, and is not contained in any other connected Abelian subgroups of G*

Example. In $U(n)$ the diagonal matrices form a maximal torus.

Theorem 1. *All maximal tori are conjugate within G. If T is a fixed maximal torus, then every conjugacy class intersects T.*

To be proved in one or two weeks. Will use Vidya's lecture.

Fix such a torus T. Consider the adjoint action of G on $\mathfrak{g}$ and then the restriction of this action to $T \subset G$.

Theorem 2. *There is a finite subset $R_+ \subset \mathfrak{t}^* \setminus 0$, $0 \neq R$ called the " positive roots" of G such that under the restriction of the adjoint action to T we have*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where T acts on $\mathfrak{t}$ trivially, where each $\mathfrak{g}_\alpha$ is two-real dimensional and can be identified with $\mathbb{C}$ in such a way that :

$$\xi \in \mathfrak{g}_\alpha, x \in \mathfrak{t} \implies Ad_{exp(x)}\xi = e^{i\alpha(x)}\xi.$$

Conversely, If the center of G is discrete, then, the data $R, \mathfrak{t}^*$ determines G up to finite covers.

XXX

should we worry about 'orientations' fo the $\mathfrak{g}_\alpha$ and choice of pos. roots ?"

Rep theory, I

Definition 2. *A representation of a group is an action of the group on a vector space by linear transformations.*

Thus: a representation of group G on a vector space V is a map $G \times V \to G$, written $(g, v) \mapsto g \cdot v$ where for each $g \in G$ the map $v \mapsto g \cdot v$ is linear on V. Equivalently, a representation of G on V is a homomorphism $\rho : G \to Gl(V)$ where we write $\rho(g)(v) = g \cdot v$. The rep is said to be over the field $\mathfrak{K}$ if V is a vector field over $\mathfrak{K}$. We will primarily work with reps over $\mathbb{C}$.

The main outlines of the theory of finite group representations over $\mathbb{C}$ was worked out by Schur in the 19th century. The main facts of the theory of compact group representations over $\mathbb{C}$ was worked out by Weyl and others in the early 20th century. The main tools are **averaging**, the notion of irreducibility, **Schur's lemma**, and and characters. One of the highlights is the Peter-Weyl theorem asserting that the full list of irreducible reps occur densely in $L^2(G)$.

Definition 3. *A representation of G on V is called irreducible if the only subspaces of V invariant under every $\rho(g)$ are the obvious ones: 0 and V.*

Two reps ρ_1, ρ_2 of G on vector spaces V_1, V_2 are called equivalent if there is an invertible G-map $A : V_1 \to V_2$intertwining them: $\rho_2(g)Av = A(\rho_1(g)(v)$ for all g, v.

For G finite there is a finite collection of inequivalent irreps. Write these as $\rho_i : G \to Gl(V_i)$ and write the dimensions of the V_i as n_i.

Theorem 3. *For a finite group G as above we have $|G| = \Sigma n_i^2$.*

The methods.

Averaging. If g is a finite group, then we can average functions on G by forming $\frac{1}{|G|}\Sigma f(g)$. If $F : G \to C$ is a function taking values in a convex set we can also average F over G.

Compact case. In the case of a compact group the role of averaging by counting is replaced by averaging by integrating.

Theorem 4. *Every compact topological group admits a unique Borel measure which is invariant under right and left translation.*

The measure of the theorem is called the Haar measure.

Proof/ Construction in the case of a compact connected Lie group G. A left-invariant volume form on G is defined by any nonzero element of $\nu \in \Lambda^n \mathfrak{g}^*$ where $n = dim(\mathfrak{g})$. Now G acts on $\Lambda^n \mathfrak{g}^*$ by sending $\theta^1 \wedge \ldots \theta^n \to (Ad^*_{g^{-1}}\theta^1 \wedge \ldots (Ad^*_{g^{-1}}\theta^n$. Since $\Lambda^n \mathfrak{g}^*$ is one (real) dimensional this action can be written as $\nu mapsto \lambda(g)\nu$ where $\nu \in \Lambda^n \mathfrak{g}^*$ is a fixed basis element as before. Now, $g \mapsto \lambda(g)$ is a Lie group homomorphism $G \to \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is the multiplicative group and so its image is a compact connected subgroup of $\mathbb{R}^*$. There is only one such: $\{1\}$. Being Ad-invariant, the volume form ν, viewed on all of G by left invariant extension, is also right invariant. Now the Haar measure is defined by normalizing ν, i.e. $\mu = \nu / \int_G \nu$.

Def. of orthogonal rep. Unitary rep. $\rho : G \to O(V)$; $\rho : G \to SO(V)$

Theorem 5. *Every complex (real) rep of a compact group on a vector space V over $\mathbb{C}$ (over $\mathbb{R}$) is unitary (is orthogonal) for some inner product on V.*

Pf: Start with any inner product $\langle \cdot, \cdot \rangle$ on V. Act on it by G by writing $\langle v, w \rangle_g = \langle gv, gw \rangle$. Now average over G, noting that the set of inner products is a convex subset of the vector space of all quadratic forms on V.

Reducibility/Irreducibility

Definition of an invariant subspace. A subspace S of rep space V is called 'invariant ' if for all $g \in G$ and all $s \in S$ we have that $\rho(g)s \in S$.

PROPOSITION 1. *Suppose that $S \subset V$ is an invariant subspace for a unitary rep ρ of cpt G. Then $S^{\perp}$ is also invariant.*

Cor. Complete reducibility of cpt group reps.

Notion of irreducible represenation. (irrep).

Countereg of complete reducibility in noncpt case: upper triangulars. Birthday rep.

Irreps in the Abelian case: all irreps 1d. (over $\mathbb{C}$) – simultaneous diagonalization.

Eg: $\hat{\mathbb{Z}}/kZ = k$th roots of unity $\subset S^1$.

Another look at theorem 1 in the finite Abelian case: $|G| = \Sigma 1^2$. Interpretation: in the Abelian case the set of unitary irreps is the same as the set of homomorphisms $G \to S^1$. These form a group under pointwise multiplication, called the dual group $\hat{G}$ to G. So the formula asserts that $|G| = |\hat{G}|$. Indeed, we have just seen that G and $\hat{G}$ are isomorphic in this case.

Circle case. Irreps of S^1: $\lambda_k(\theta) = e^{ik\theta}$.

Q: How do we know this is the full list of irreps for the circle??

Schur's lemma. A

V, W irreducible reps of G. $F; V \to W$ a G-map. Then either $F = 0$ or F is invertible.

Proof. $kerF$ is an invariant subspace of V so is either 0 or V. $im(F)$ is an invariant subspace of W so is either 0 or W. If $im(F) = 0$ then $F = 0$. If $im(F) = W$, $W \neq 0$ and $V \neq 0$ then $ker(F) \neq V$ so $ker(F) = 0$ which is to say that F is an isomorphism.

Schur's lemma. B

If V is an irrep of G and $L : V \to V$ is a G-map, then $L = \lambda I$ for some complex number λ.

Proof. If $L = 0$ then $\lambda = 0$. Otherwise, consider $L - \lambda I$. It has a nontrivial kernel for some λ. By the Schur Lemma, this kernel must be all of V so that $L = \lambda I$.

QED

Breaking any rep up into irreps.

Functorialites on reps.

If V, W are reps of G, so are V^*, $V \oplus W$, $\Lambda^k V$, $S^k(V)$, $V \otimes W$ and $Hom(V, W) = V^* \otimes W$.

Notation. For m a positive integer and V a rep, write $mV = V \oplus V \oplus \ldots \oplus V$ (m-times.)

THEOREM 6. *Let $\rho : G \to V$ be an finite dim. irrep. of compact G Then there exist irreps V_i of G and multiplicities m_i, positive integers such that $V = \Sigma m_i V_i$ as G-spaces. This decomposition is unique.*

Proof. Knock off one V_i at a time. Use induction on dimension, together with the fact that the orthogonal complement of an invariant subspace is invariant.

For uniqueness of the decomposition we could use Schur in its "delta-function" type guise of $Hom_G(V_i, V_j) = \delta_{ij} I$ where $V_1, V_2, \ldots, V_k$ are a list of inequivalent irreps and where $Hom_G(V, W)$ is the space of linear G-maps $V \to W$, for two given reps V, W.

QED

For compact Lie group G of dimension greater than 0 (i.e not finite) the list of irreps is countably infinite.

BASIC EGS. The circle. $\hat{G} = \mathbb{Z}$.

The group $SU(2)$. Let V_{k+1} be the space of homogenous degree k complex polynomials on $\mathbb{C}^2$. It is a complex vector space of dimension $k+1$. Now $SU(2)$ acts on $\mathbb{C}^2$ [birthday rep], which we write as $z \mapsto gz$. Then it acts on V_k by composition: $(gP)(z) = P(g^{-1}z)$.

THEOREM 7. . *The V_k are the complete list of irreps for $SU(2)$.*

We have $SO(3) = SU(2)/\{\pm I\}$ so that if $\rho(-I) = 1$ then ρ is a rep of $SO(3)$. When the degree of the polynomial is even we have $\rho_k(-1) = I$.

THEOREM 8. *The V_k, for $SU(2)$ with k odd (so degree even) are a complete list of irreps for $SO(3)$*

PROOF: TO APPEAR ...

Exercises.

$SO(3)$ acts on $\mathbb{R}^3$, hence on $\mathbb{R}^3 \otimes \mathbb{R}^3$. Decompose $\mathbb{R}^3 \otimes \mathbb{R}^3$ into its $SO(3)$-irreps.

G acts on the space of functions on G. If G is finite, we can identify this space of functions with the "group algebra" $\mathbb{C}G$ where $\delta_g(h) = 0, h \neq g$ and $\delta_g = g$. An element of $\mathbb{C}G$ is then written $\Sigma c(g)\delta_g$. If G is compact but infinite, then we usually use $L^2(G)$. There is the right and left regular multiplication: $L_g f(h) = f(g^{-1}h)$ and $R_g f(h) = f(gh)$ ....

Theorem. The group algebra decomposes under $G \times G$ into

$L^2(S^1)$. Breaking into irreps. Fourier coefficients. (Peter-Weyl for the circle.)