Universal cover of a Lie group.

Last time Andrew Marshall introduced the notion of homotopy, the fundamental group, and covering space. We recall : a covering space is a fiber bundle with discrete fiber. The fundamental group of a topological space is a group whose definition depends on the choice of a base point $m_0 \in M$. This group is denoted by $\pi_1(M, m_0)$ When M is a connected manifold, then the $\pi_1(M, m)$'s at different points m are isomorphic, so we speak of *the* fundamental group of M.

A morphism of coverings $p_1 : M_1 \to M, p_2 : M_2 \to M$ is a covering $f : M_1 \to M_2$ such that

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & M & \end{array}$$

commutes.

Theorem 1. *(1) Let M be a connected manifold. Then every covering space of M with countable fiber is a manifold in such a way that the covering map is a smooth submersion.*

(2) Let G be a connected Lie group. Then every connected covering space of G is a Lie group in such a way that the covering map $p : G_1 \to G$ is a Lie group homomorphism. Moreover $ker(p)$ lies in the center of G_1.

Definition 1. *A connected topological space is said to be 'simply connected' if its fundamental group is trivial (the 0 group). Equivalently, a simply connected topological space has the property that every loop is contractible. Or that every loop bounds a disc.*

Theorem 2. *Let M be a connected manifold. Then there is a connected covering space of M called the "universal cover" and denoted $\tilde{M}$, which is unique up to covering isomorphism (over M) and is characterized by any one of the following properties.*

- *$\tilde{M}$ is simply connected*
- *$\tilde{M}$ is an initial object in the category of connected coverings of M*
- *$\tilde{M}$ is a principal $\pi_1(M)$- bundle.*

A few words are probably in order regarding item (2). We can form a category whose objects are connected covering spaces of M and whose morphisms are covering morphisms as defined above. An 'initial object' is one such that given any other object there is a unique morphism from the initial object to the other object. Initial objects (or final objects) in categories are always uniquely defined up to isomorphism.

By theorem 1, the universal cover of a connected Lie group is a simply connected Lie group which covers G by a LIe group homomorphism. Then the projection $p : \tilde{G} \to G$ is a Lie group homomorphism with discrete normal kernel $ker(p)$. By item(3) or theorem 1, this kernel is isomorphic to the fundamental group $\pi_1(G)$. By the exercise of last week regarding discrete normal subgroups, $ker(p)$ lies in the center of G. As a corollary then:

Corollary 1. *The fundamental group of a Lie group is Abelian.*

Examples. $\mathbb{R} \to S^1$; $x \mapsto exp(i2\pi x)$ realized $\mathbb{R} = \tilde{S}^1$. The kernel of this map is $\mathbb{Z}$.

$SU(2) \to SO(3)$ by the adjoint rep, or via the quaternionic exercise, realizes $SU(2) = S^3$ as $\tilde{S}O(3)$. The kernel of this map is the two element group $\mathbb{Z}/2\mathbb{Z}$.

Proof of theorem 1. (1). If $f : M_1 \to M$ is a covering map and $n \in M_1$, take a nbhd U of $m = f(n) \in M_1$ which is evenly covered. We can take a smaller nbhd $V \subset U$ containing m which is a coord nbhd with coords $x : V \to \mathbb{R}^n$. Then $f^{-1}(V)$ is homeomorphic to a disjoint union of copies of V precisely one of which contains n. Call this V'. Then the restriction of $x \circ f$ to V' forms coord on M_1. We leave it to the reader to check smoothness of overlaps and that f, with this def. is a smooth submersion.

(2) By (1), we have that G_1 is a smooth manifold. The group law will have to wait until theorem 2.

Proof of theorem 2.

I will start with the notion of the "fundamental groupoid" of a manifold. See Weinstein, in the Notices: for more about groupoids.

Consider the space $Gr(M) = P(M)/\sim$ consisting of all smooth paths $\gamma : [0,1] \to M$ modulo the equivalence relation of endpoint fixing homotopies. We can compose some elements of the groupoid, but, unlike a group, not all elements can be composed. Specifically, we can compose two elements if and only if the enpoint of the first coincides with the initial point of the second. We formalize this with symbols. Because the homotopies used in defininig $\sim$ are endpoint-fixing, there are well defined maps $\alpha : Gr(M) \to M$ and $\beta : Gr(M) \to M$ which assign to a homotopy class of paths its initial point or final point. Then gh is defined if and only if $\alpha(h) = \beta(g)$, in which case $gh = [\gamma_g * \gamma_h]$ where $g = [\gamma_g], h = [\gamma_h]$.

Aside A groupoid is essentially a group where not all elements can be composed. It is endowed with two maps α, β "source" and "target' which formalize which elements can be composed. I leave for read Weinstein for a full definition.

Together the maps (α, β) make $Gr(M)$ into a covering space of $M \times M$.

For our purposes, it will be simpler to fix the initial point, thus considering the submanifold $\tilde{M} = \alpha^{-1}(m_0) \subset Group(M)$. We will show that $\beta : \tilde{M} \to M$ does indeed give $\tilde{M}$ the structure of a universal cover.

$\tilde{M}$ covers: Let $[x, y]$ denote the line segment joining x to y, two points in a vector space. If we restrict x, y to lie in a convex domain, such as a ball, then $[x, y]$ varies smoothly with x, y. Cover M with coord nbhds homeo to balls, and write $\psi_a : U_a \to B$ for the coord charts. For x, y in any one of these nbhds U_a , use $[x, y]$ to denote the smooth curve $\psi_a^{-1}[\psi_a(x), \psi_a(y)]$ in $U_a \subset M$. Now, suppose $p = [\gamma] \in \tilde{M}$. with $x = \beta(p) = \gamma(1) \in U_a$. Then $[x, y] * \gamma$ defines a piecewise smooth path joining m_0 to y and this path varies smoothly with y. Thus $s(y) = [[x, y] * \gamma]$ is a local section of β over U_a, i.e $\beta \circ s = I_{U_a}$. The image of s is a nbhd of p in $\tilde{M}$. Now, let $[\gamma_j]$ vary over all points of $\beta^{-1}(x)$. By this same construction, we form an even cover of U_a : $\beta^{-1}(U_a) \cong \cup_i U_{a,i}$ where $U_{a,i} = s_i(U_a)$.

Universal covering property. Use theory of homotopy lifting. See Greenberg, or Duistermaat.

Special case of $M = G$. Take $m_0 = e$, the identity. Multiplication on $\tilde{G}$: $[\gamma_1][\gamma_2] = [\gamma_3]$ where $\gamma_3(t) = \gamma_1(t)\gamma_2(t)$. We get a Lie group.

...

XXX

Usefullness of the notion of the universal cover of a group. Reasons for formulating.

1) Not every Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{h}$ extends to a Lie group homomorphism $G \to H$. But if G is connected and simply connected then it is true that every such Lie algebra homomorphism extends.

2) For any Lie algebra $\mathfrak{g}$ there is a UNIQUE (up to Lie group isomorphism) connected, simply connected Lie group whose Lie algebra is G.

3) If $\tilde{G}$ is the universal cover of G then the every representation of G comes from a representation of $\tilde{G}$.

4) Suppose that G acts on a manifold M and that $E \to M$ is a fiber bundle over M. The action of G may not lift to an action on E. But the action of $\tilde{G}$ via $\pi : \tilde{G} \to G$ does extend to an action on E¿

5) If G is compact, connected with finite fundamental group then there are a finite number of compact connected groups G' with $Lie(G') = Lie(G)$. They form a poset where $G_1 < G_2$ if there is a Lie homomorphism $G_2 \to G_1$. There is a unique largest element, $\tilde{G}$, which is the universal cover of any one of these groups. There is a unique smallest element, $Ad(G)$ which is the image of any one of them under the adjoint rep. For $\tilde{G}$ we have $\pi_1(\tilde{G}) = 0$ and $Z(\tilde{G}) = \Gamma$ while for $Ad(G)$ we have $\pi_1(Ad(G)) = \Gamma$ and $Z(Ad(G)) = 0$¿