Lie groups from the dual point of view.

I am giving this lecture for two reasons. One ; the appearance of Jair Koiller as a guest lecturer on Thursday, Feb 22 or 24. Two: its importance for the theory of connections on principal bundles, which Alex, Don, and Vidya should learn.

Background. Cartan did most of his work using one-forms rather than vector fields. For him, the most natural way to understand the structure of a Lie group was not from its Lie algebra, but rather from the dual of its LIe algebra.

$\mathfrak{g}$ Basis $E_1 \dots, E_n$ Structure constants C^k_{ij} defined by

$$[E_i, E_j] = \Sigma c^k_{ij} E_k$$

Geometric interpretation. The E_i are left-invariant vector fields on G. Any left invariant vector field has the form $\Sigma x^i E_i$ for some constants x^i. The E_i also *frame* G, or equivalently, form a "parallelization" of G.

$\mathfrak{g}^*$ = dual space to $\mathfrak{g}$. Dual basis $\theta^1, \dots, \theta^n$ so that

(1) $$\theta^i(E_j) = \delta^i_j$$

$$d\theta^i = \Sigma_{j<k} c^i_{jk} \theta^j \wedge \theta^k.$$

Geometric interpretation. The θ^i are left-invariant one-forms on G. Any left invariant one-form has the form $\Sigma p_i \theta^i$ for some constants x^i. The θ^i also form a *co-frame* for G.

Proof of formula for $d\theta^i$.

We use a formula due to Cartann which related the exterior differential and the Lie bracket. This formula is

$$d\theta(X, Y) = X[\theta(Y)] - Y[\theta(X)] - \theta([X, Y])$$

and is valid on any smooth manifold, with θ a one-form on the manifold, X, Y vector fields. Applied to θ^i with $X, Y = E_j, E_k$ we find

$$d\theta^i(E_j, E_k) = -\theta^i([E_j, E_k])$$

where we have used the constancy of $\theta^i(E_j)$ and $\theta^i(E_k)$ (see eq (1)). Now use eq (**??**) to get

$$d\theta^i(E_j, E_k) = -c^i_{jk}$$

The structure equation can also be writte

$$d\theta^i = \frac{1}{2} \Sigma c^i_{jk} \theta^j \wedge \theta^k.$$

where the sum is now over all pairs j, k.

Example $G = SO(3)$. $\theta^1, \theta^2, \theta^3$ dual to the standard basis for $\mathfrak{g}$ (rotation about three orthog axis.) One finds:

$$d\theta^i = \frac{1}{2} \epsilon j k^i \theta^j \wedge \theta^k$$

where $\epsilon j k^i = \epsilon_{ijk}$ is the usual "completely alternating symbol" defined variously by:

$$\epsilon_{ijk} = \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{jik}$$

and $\epsilon_{123} = 1$ or by

$$det(X) = \Sigma \epsilon_{ijk} X^1_i X^2_j X^3_k$$

where the entries of the 3×3 matrix X are X^I_j.

Example. G Abelian $\iff c^k_{ij} = 0$.

Maurer-Cartan form. This is spectacular but simple one-form is canonically attached to any Lie group called the Maurer-Cartan form. It is not a standard one-form, but rather a $\mathfrak{g}$-valued one-form.

Def. If V is a vector space and M a manifold, then a V-valued one-form is a collection of smooth maps $T_m M \to V$. In other words, it is a smooth section of $T^*M \otimes V$.

The Maurer-Cartan form is really just a methodical way of trivializing the tangent bundle of the Lie group. Recall that L_g induces isomorphisms $\mathfrak{g} = T_e G \to T_g G$. So we have a globally defined $\mathfrak{g}$-valued one-form Θ on G obtained by setting

$$\Theta_g(V_g) = (dL_g)^{-1} V_g.$$

If G is a matrix group, so that $G \subset Gl(V)$ for some vector space V, then:

$$\Theta = (dg)g^{-1}$$

where $dg : T_g G \to \mathfrak{g}l(V)$ is the inclusion. **Proof.** At $g = Id$, $dg : T_e G \to gl(V)$ is the realization of $\mathfrak{g} = T_e G$ as a vector space of matrices. Now observe that $(dg)g^{-1}$ is left-invariant.

An alternative, equivalent def of Θ is that

$$\Theta(X^L) = X(\textbf{??}) \tag{2}$$

where X^L is any left-invariant vector field, and $X = X^L(e) \in \mathfrak{g} = T_e G$ is its value at the identity.

Structure formula:

$$d\Theta = \frac{-1}{2} \Theta \wedge \Theta$$

Proof. Cartan's formula applies to vector-valued one-forms. Thus:

$$d\theta(X^L, Y^L) = 0 - 0 - \Theta([X^L, Y^L])$$

by eq (**??**), But $[X^L, Y^L] = [X, Y]^L$ and $X = \Theta(X^L), Y = \Theta(Y^L)$ so that

$$d\Theta(X^L, Y^L) = [\Theta(X^L), \Theta(Y^L)])$$

This looks like we have just proved that $d\Theta = [\Theta, \Theta]$ and by one definition of the bracket of one-forms we have. What about the pesky $1/2$ in the formula? It is a matter of notation, and comes from the fact that if α, β are $\mathfrak{g}$-valued one-forms (on any manifold) then we want $\alpha \wedge \beta$ to also be a $\mathfrak{g}$-valued two-form. Define it by:

$$\alpha \wedge \beta(X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]$$

then:

$$[\Theta(X^L), \Theta(Y^L)] = \frac{1}{2} \Theta \wedge \Theta(X^L, Y^L)$$

and the formula is proved.

Relation to connnections on principal bundles.

The Maurer-Cartan form is basic to understanding (1) connections and (2) Cartan's approach to differential geometry.

Connections on principal bundles. In the theory of connections on principal bundles, the "connection one-form" is a G-valued one-form, usually written A, defined on a bundle of Lie groups. Restricted to each fiber it equals the Maurer-Cartan connection on that fiber. In that theory, the analogue of changing coordinates is changing 'local trivializations" or, from the active point of view, making a gauge transformation. In local coordinates on the base manifold, a change of local trivialization is given by G valued function on the overlap $U_\alpha \cap U_\beta$ of two open sets; write this function as $g : U_\alpha \cap U_\beta \to G$. Then, the transformed connection one-form $\tilde{A}$ satisfies the transformation rule:

$$\tilde{A} = g^{-1}Ag + g^{-1}(dg) \qquad (*R).$$

in case G is a matrix group Realizing that $g^{-1}(dg)$ is the matrix group expression for the Maurer-Cartan form, we see that an equivalent way to write this formula is

$$(A)_\alpha = Ad_{g^{-1}}A_\alpha + \Theta$$

where $(A)_\beta, A_\alpha$ are the connection forms in the two local trivializations. This is the form you will find in Kobayashi-Nomizu, vol. 1, ch. 2.

Notational caveat. In many places one will find instead the different transformation rule

$$\tilde{A} = gAg^{-1} + (dg)g^{-1} \qquad (*L)$$

This is the formula you will find in Chern's Complex Manifolds without Potential Theory (a beautiful choice for learning about connections) The reason behind the difference in the formulae (*R) and (*L) is that of *right* vs *left* principal bundles. For KN (*R) the group G acts on the PBs on the right, while in (*R), the group acts on the left. In (*L) the form $g^{-1}(dg)$ appearing is the RIGHT-INVARIANT Maurer-Cartan on a matrix group.

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Derivations of formulae; getting signs straight.

When the pb is the (o.n.) frame bundle of a vector bundle $E \to M$ w connection ∇ then the connection form A can be defined locally by the formula

$$\nabla s = sA$$

In this formula s is a local frame, so that $s = (s_1, \ldots, s_r)$ where the s_a are local sections of $E \to M$ such that $s(x)$ is a (o.n.) basis for E_x. ∇s denotes the r-vector of E-valued one-forms whose ath component is $X \mapsto \nabla_X s_a$ and sA is the r-vector of E-valued one-forms the E-valued one-form $X \mapsto \Sigma s_a A^a_b(X)$.

Now connections on vector bundles satisfy

$$\nabla f\psi = df \otimes \psi + f\nabla\psi$$

where f is any smooth function on the base M and $\psi : ME$ is any local section. It follows from this that if we change frames according to:

$$\tilde{s} = sg$$

which means that $\tilde{s}_a = \Sigma s_b g^b_a$ then

$$\tilde{s} = (\nabla s)g + s \otimes dg = (sA)g + sdg = \tilde{s}(g^{-1}Ag + g^{-1}dg)$$

which is (*R). If instead we had written $\tilde{s} = gs$ we would have derived (*L).

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More on (*L) vs (*R); left vs right MC form appearing... A inverts $\sigma_q : \mathfrak{g} \to V_q$ In a loc triv, and in the case where G acts on the pb on the RIGHT we have $\sigma_q(\xi) = (x, g\xi) = (x, \xi^L)$ which implies that A, restricted to fibers, must be the LEFT-INV MC one-form on the fibers: $A(\sigma_q\xi) = \xi$ reading $A_{x,g}(0 \oplus \xi^L) = \xi^L$ in a loc. triv.

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The curvature of A can be defined as

$$F_A = dA + \frac{1}{2}[A, A]$$

The addition of $\frac{1}{2}[A, A]$ is expressly chosen so that in the case of $A = \Theta$ we find $F_A = 0$.

Cartan's moving frame approach to differential geometry.

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Mechanics. Symplectic geometry. Co-adjoint orbits.