The Nilpotent case.

A Lie algebra is called "nilpotent" if there is an integer such that all brackets of length $[X_1, [X_2, \ldots X_\ell] \ldots$ of length $\ell > r$ are zero. Here is an alternative definition. Write $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] = span\{[X, Y] : X, Y \in \mathfrak{g}\}$ and inductively $\mathfrak{g}^{s+1} = span[\mathfrak{g}, \mathfrak{g}^s]$. Then $\mathfrak{g} \supset \mathfrak{g}^2 \supset \ldots \supset \mathfrak{g}^s \supset \mathfrak{g}^{s+1} \supset \ldots \mathfrak{g}$ forms a decreasing set of ideals within $\mathfrak{g}$. (Eventually it stabilizes: $\mathfrak{g}^s = \mathfrak{g}^{s+1}$.) If for some r we have $\mathfrak{g}^{r+1} = 0$ then $\mathfrak{g}$ is nilpotent. The smallest integer r such that vanishing occurs is called the 'step' of the algebra.

Definition 1. *A Lie group is called nilpotent if it is connected and its Lie algebra is nilpotent.*

Examples. Connected abelian Lie groups are nilpotent of step 1.

The Heisenberg group is nilpotent of step 2. It can be realized as upper triangular matrices with 1's on the diagonal:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

The group of all $n \times n$ upper triangular matrices with 1's on the diagonal is nilpotent of degree $n - 1$.

Motivations for studying nilpotent groups.

They arise in the theory of distributions (subbundles of the tangent bundle), and hence in nonlinear control theory.

They are the first case where Kirrilov's orbit method in representation theory worked. (The orbit method takes literally the quantum to classical correspondence. To build a representation of a group on a Hilbert space it begins by finding a "classical mechanical system" on which that group acts, and constructs the representation as a "quantization" of that classical system.)

They are a key ingredient in Gromov's theory of "almost flat Riemannian manifolds".

The main goal of this note is to prove the following characterization of nilpotent groups.

Theorem 1. *If G is a nilpotent group then the exponential map $\mathfrak{g} \to G$ is onto, and has no singular points.*

Corollary 1. *If $\tilde{G}$ is a simply connected nilpotent Lie group then it is isomorphic to $\mathbb{R}^n$ with a multiplication law whose identity is 0, whose inverse is $X \mapsto -X$ and whose multiplication law $X, Y \mapsto m(X, Y)$ is polynomical*

The proof of the theorem is based two observations

(1) If r is the step of a nilpotent Lie algebra, then $\mathfrak{g}^r \subset \mathfrak{z}(\mathfrak{g})$ and is not empty. (2) the center of a Lie group is closed, and hence is an Abelian Lie group in its own right.

Proof of theorem. By induction on the dimension of $\mathfrak{g}$. If $dim(\mathfrak{g}) = 1$ then the algebra is Abelian, and the Abelian case has been established. ($G = \mathbb{R}^1$ or S^1.) (One can also show that any nilpotent Lie algebra of dimension 2 is Abelian, but this is not necessary for the proof.)

Make the inductive hypothesis that the theorem holds for all nilpotent groups of dimension $\leq n$. Let G be a nilpotent group of dimension $n+1$. Let $Z^0 \subset G$ be the identity component of its center. By the above observations, this is a connected Abelian Lie group of dimension greater than 0. Now, the exponential map is onto in Abelian Lie groups. And any Abelian Lie group of dimension ℓ has the form $T^k \times \mathbb{R}^{\ell-k}$. We can choose a **closed** one-parameter subgroup $H = \{exp(tz) : t \in \mathbb{R}\} \subset Z^0$ of Z^0. (If Z^0 is compact, hence a torus, then the z must generate one of the circles of $T^n = S^1 \times \ldots S^1$.) Now H is closed and normal, consequently we can form the quotient group G/H.

G/H satisfies the inductive hypothesis. It is a Lie group of dimension $n = dim(G) - dim(H)$. It is connected, being the continuous image of the connected group G. It is nilpotent since its Lie algebra $\mathfrak{g}/\mathfrak{h}$ is nilpotent. By the inductive hypothesis, the exponential map $\mathfrak{g}/\mathfrak{h} \to G/H$ is onto.

Consider the diagram :

$$\begin{array}{ccc} \mathfrak{g} & \to^{exp} & G \\ \downarrow & & \downarrow \\ \mathfrak{g}/\mathfrak{h} & \to^{exp} & G/H) \end{array}$$

This diagram commutes. The bottom arrow and the left arrow are onto. Thus $\pi \circ exp : \mathfrak{g} \to G/H$ is onto.

I claim that it follows from this that $exp : \mathfrak{g} \to G$ is onto. For let $g \in G$ be any element. Since $\pi \circ exp$ is onto, exp hits the coset through g. In other words, there is an $h = exp(tz) \in H$ and an $X \in \mathfrak{g}$ such that $gh = exp(X)$. But then $g = exp(X)exp(-tz)$. And from the observation on central elements, made at the end of the 'Abelian' note, we have that $g = exp(X - tz)$.

This establishes that exp is onto.

To see that its differential is everywhere onto, note that since $A = ad_X$ is nilpotent for each $X \in \mathfrak{g}$ we have that $spec(ad_X) = \{0\}$. And since $F(0) = 1$ we have that $F(ad_X)$ is invertible, and hence $dexp_X$ is invertible, according to the formula for ad_X.

QED