The Abelian case.

Following Adams, lemma 2.18, theorem 2.19.

Theorem. If G is a connected Abelian Lie group of dimension n then $G = T^r \times \mathbb{R}^{n-r}$ for some nonnegative integers r.

The proof uses the very rough form of the CBH formula discussed in class: $exp(X)exp(Y) = exp(X+Y+O(|X|^2+|Y|^2))$. (This is Adams, lemma 2.18.)

Proof of theorem: Let G be connected, Abelian and N a large integer. Then

$$\begin{aligned} exp(X)exp(Y) &= exp(X/N)^N exp(Y/N)^N \\ &= [exp(X/N)exp(Y/N)]^N \\ &= exp(X/N+Y/N+O(1/N^2))^N \\ &= exp(X+Y+O(1/N)) \end{aligned}$$

We used the fact that $exp(X/N)$ and $exp(Y/N)$ commute in going to the second line. The other equalities are valid for a general Lie group. Letting $N \to \infty$ in this equality we get that

$$exp(X)exp(Y) = exp(X+Y)$$

which asserts that exp is a Lie group homomorphism from $(\mathfrak{g}, +)$ to G. It follows that (1) the image of exp is an open subgroup of G, and hence all of G, and that (2) the kernel K of exp is a discrete subgroup of $\mathfrak{g}$. (It is discrete because exp is a local diffeo near 0, and using translation, near any point. Now the discrete subgroup of a real finite dimensional vector space (viewed as a group under +) is a free Abelian group generated by r linearly independent vectors. Consequently $G = \mathfrak{g}/K \cong \mathbb{R}^r/\mathbb{Z}^r \times \mathbb{R}^{n-r} = T^r \times \mathbb{R}^{n-r}$ where the $\mathbb{R}^r$ is the span of these r vectors.

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A central observation It follows from the calculation involving $X/N, Y/N$ above that if $z \in \mathfrak{g}$ is central, and $X \in \mathfrak{g}$ is any element, then $exp(X+z) = exp(X)exp(z)$.

Other loose ends – but IMPORTANT:

Theorem 1. *Every closed subgroup of a Lie group is Lie group in its own right. In particular, it is an embedded submanifold.*

Theorem 2. *Let G be a Lie group with algebra $\mathfrak{g}$. Then associated to each Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a unique* **immersed** *connected Lie subgroup $H \to G$ whose differential realizes $\mathfrak{h}$.*

WARNING. The H of the theorem might not be closed and hence might not be embedded, viz. the irrational line on the torus.

I will not prove the closed subgroup theorem. See Adams for a proof.

I will sketch the subalgebra to subgroup theorem. This proof is an application of the Frobenius theorem from manifolds 1.