II. Shape space. Reduction by rotation. Central configurations. McGehee blow-up.

A. Shape sphere. Pictures for the three-body problem. Shape space. The idea of a Riemannian submersion. The symplectic quotient at zero angular momentum as the (co)tangent bundle of shape space.

B. McGehee blow-up. Collision flow. Regularizing triple collision. Moeckel's bifurcation cartoons for the planar 3 body problem: REMAINS TO INCLUDE. Sundman total collision implies zero ang mom theorem:: REMAINS TO INCLUDE . Chenciner perspective (?? INCLUDE ??).

Lecture 2.

1. Shape space.

II. The shape space for the three body problem is the space whose points represent (oriented) congruence classes of triangles. By the SSS theorem of Euclid, this shape space is three dimensional. Indeed, the three side lengths, r_{12}, r_{23}, r_{31} are functions on shape space and for the spatial three-body problem they are a complete system of invariants: these three lengths uniquely determine the oriented congruence class of the triangle. In the case of the planar three-body problem, we will need one more variable corresponding to orientation. For unlike space, in the plane triangles with vertices labelled have orientations. There will be precisely two oriented congruence classes of planar triangles with given side lengths. For example, we have two Lagrange triangles: two different equilateral triangles in the plane. A good orientation varialbe is the triangle's *signed area*

$$\Delta = \frac{1}{2}(q_2 - q_1) \wedge (q_3 - q_1).$$

We have seen that there are two basic functions for the N-body problem, the (negative) potential U and the kinetic energy K. As far as the potential U is concerned, the coordinates r_{ij} are perfect: the expression for U could not be simpler. However, the expression for K in terms of the r_{ij} is horrible, and cannot even be written down without first specifying the total angular momentum J. For K it is better to use other variables. I have found it best to use variables adapted to K, and re-express U in their terms.

Good shape variables for K are r, ϕ, θ where (ϕ, θ) are standard coordinates on the sphere, arranged so that the usual equator is given by $\phi = 0$. The variable r is always $\sqrt{I}$ so that it measures the overall size of the triangle. Thus

$$r^2 = I = \frac{1}{m}(m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2)$$

is the total moment of inertia. The relation between r, ϕ, θ and our previous shape functions r_{ij}, Δ are:

$$r^2 \sin(\phi) = \Delta$$

and

$$r_{ij}^2 = \frac{m_i + m_j}{2m_i m_j} r^2 (1 - \gamma_k(\theta)\cos(\phi)),$$

with i, j, k any permutation of $1, 2, 3$ and with

$$\gamma_k(\theta) = \cos(\theta_k^0 - \theta).$$

for three fixed angles θ_k^0 whose values depend on the masses. The locus $\phi = 0$ corresponds to the set of collinear triangles. On this collinar locus, the three angles $\theta = \theta_k^0$ indicate the locations of the three binary collisions.

Here is promised simple K in these coordinates.

$$K = \frac{1}{2}\{(\dot{r}^2 + \frac{r^2}{4}ds^2_{sphere}) + \frac{J^2}{r^2}\} \tag{1}$$

where

$$ds^2_{sphere} = \dot{\phi}^2 + \cos^2(\phi)\dot{\theta}^2,$$

the standard metric on the unit sphere, and where J is the total angular momentum.

PICTURE HERE. OF SHAPE SPHERE.

We will derive and generalize formula (1) in the next subsection.

1.1. **Shape spaces, generally.** Consider a Riemannian manifold Q on which a Lie group G acts by isometries. Then the shape space for (Q, G) is, by definition, the quotient space

$$S = Q/G.$$

We assume the G-action is 'nice' so that the quotient is a Hausdorff space. (Our action is nice.) The shape space inherits a metric d_S from Q's distance function d_Q according to: the distance between two points $s_1, s_2 \in S$ satisfies:

$$d_S(s_1, s_2) = inf_{q_1, q_2 : \pi(q_1) = s_1, \pi(q_2) = s_2} d_Q(q_1, q_2).$$

where

$$\pi : Q \to Q/G = S$$

is the quotient map.

Recall that a G-action is called "free" if $gq = q$ implies that $g = id.$. It is called proper if the map $G \times Q \to Q$ is proper. If the G-action is free and proper then shape space is a manifold, and the quotient map is a submersion. In the case of free and proper actions, the metric d_S comes from a Riemannian metric. and the shape space metric comes from a Riemannian metric on S induced from that on Q.

This quotient Riemannian metric is succinctly described via the notion of a *Riemannian submersion* Let Q, S be Riemannian manifolds and $\pi : Q \to S$ a submersion.

Definition 1. *The submersion π is called a Riemannian submersion if for each $q \in Q$ the restriction of $d\pi_q$ to $ker(d\pi_q)^{\perp}$ is a (linear) isometry onto $T_{\pi(q)}S$.*

Exercise 1. *If Q alone is given a Riemannian structure, then a submersion $\pi : Q \to S$ induces a unique Riemannian structure on S such that the map π is a Riemannian submersion.*

Example. When G acts freely and properly on the Riemannian metric space Q by isometries, then the shape space Q/G inherits a unique Riemannian metric making the quotient map into a Riemannian submersion. Moreover the infimums for the formula for the distance d_S are realized so the 'inf' can be replaced by a 'min'.

Main Example. Let $V = \mathbb{C}^D$ be the standard complex D-dimensional Hermitian vector space and take $Q = S^{2D-1} \subset V$ to be the unit sphere within V with its induced metric. Take for G the group $G = S^1$ of unit complex numbers acting on V by complex scalar multiplication. This action is free if we delete the origin of V, and it maps the sphere $S^{2D-1} \subset V$. The shape space S^{2D-1}/S^1 is diffeomorphic to the complex projective space $\mathbb{P}(V) = \mathbb{CP}^{D-1} = V \setminus \{0\}/\mathbb{C}^*$. The quotient map $\pi : S^{2D-1} \to \mathbb{CP}^{D-1}$ is the Hopf fibration. Declaring this projection to be a Riemannian submersion endows $\mathbb{CP}^{D-1}$ with its standard Fubini-Study metric. In the special case $D = 2$, the quotient map is the standard Hopf fibration $S^3 \to S^2 = \mathbb{CP}^1$.

Proposition 1. *If $V = \mathbb{C}^D$ is a Hermitian vector space of complex dimension D then V/S^1 is isometric to the metric cone over $\mathbb{CP}^{D-1}$.*

We best pause to recall the notions of 'topological cone' and 'metric cone' over a Riemannian manifold X. The topological cone over X is obtained by forming the product space $[0, \infty) \times X$ of X with the ray $[0, \infty)$ and then crushing (identifying) $0 \times X$ to a single point called the cone point. The metric structure on this cone is that given by the metric $dr^2 + r^2 d^2 s_X$ where $r \geq 0$ parameterizes the ray, and where ds_X^2 is the Riemannian metric on X. (For the more general case where X is a length space, see Gromov XXX for the definition of X's metric cone.)

Exercise 2. *Show that the cone over the unit sphere $S^{N-1} \subset \mathbb{R}^N$ is $\mathbb{R}^N$.*

Exercise 3. *Find the radius r_0 of the circle such that the standard positive cone $x^2 + y^2 - x^2 = 0, z \geq 0$ is isometric to $Cone(S^1(r_0))$.*

Proposition 2. *Let V be a D Euclidean vector space and G a group of isometries of V acting linearly on V, freely away from 0. Set $X = S^{D-1} \subset E$ be the unit sphere in E. Then $V/G \cong Cone(X/G)$ as metric spaces, where X/G is given the shape space metric.*

Proof of prop 2 Expressed in spherical coordinates, E has metric $dr^2 + r^2 d^2 s_X$. The G action, being by isometries, and leaving 0 fixed leaves the radial function

$r = dist(0, q)$ invariant. Thus the quotient metric is, formally, $dr^2 + r^2 d^2 s_{X/G}$, which is the metric for the cone over X/G.

Prop 1 now follows immediately from prop 2 and the discussion above.

The case of propositon 1 is precisely that of the planar N-body problem. Identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard manner. Then rotation becomes complex scalar multiplication by a unit complex number, while a homothety with scale factor k becomes scalar multiplication by $ke^{i\theta}$. The configuration space for the problem is the Hermitian vector space $\mathbb{C}^N$, endowed with its mass metric. Dividing this space by translations, we get $\mathbb{C}^N/\mathbb{C}$ isometric to the linear subspace $\mathbb{C}^{N-1} \subset \mathbb{C}^N$ consisting of those configurations with center of mass at the origin, endowed with the Hermitian induced metric. Take this $\mathbb{C}^{N-1}$ as our V in proposition 1. The group of rotation acts on this V by scalar multiplication by unit modulus complex numbers.

We have seen that the shape space for the planar N-body problem is the cone over $\mathbb{CP}^{N-2}$.

When $N = 3$, we have $\mathbb{CP}^{N-2} = \mathbb{CP}^1 = S^2(1/2)$- the sphere of radius 1/2. The 1/2 comes about because in any great circle in $S^3 \subset \mathbb{C}^2$ the points diametrically opposed are antipodal: (z_1, z_2) and $(-z_1, -z_2)$ are as far apart as they can get on the sphere: they are distance π apart on the sphere, and along any great circle connecting them. But because $-1 = exp(i\pi)$ these two antipodal points represent the same point in $\mathbb{CP}^1$. Consequently 1/2 an arc of a great circle in S^3 projects to a full great circle – once around $\mathbb{CP}^1$. This 1/2 is the 1/2 of $\mathbb{CP}^1 = S^2(1/2)$. The term $\frac{1}{r} ds^2_{sphere}$ of formula 1 is the standard metric on $S^2(1/2)$. Hence, if $J = 0$ so that the term J^2/r^2 were to vanish, the expression there is the expression for the shape space metric for the planar 3-body problem.

The cone point corresponds to triple collision. The triple collision point – the origin in center of mass coordinates – is the only point of $V = \mathbb{C}^2$ where where the S^1 action fails to be free.

2. Reduction at zero momentum. Notion of Riemannian submersion

We now explain the appearance of J in the metric formula 1.

We return to generalities: G acting on a Riemannian Q by isometries. We have the lifted action of G on TQ and on T^*Q. We recall the formula for the momentum map for this lifted G action on T^*Q:

$$J : T^*Q \to \mathfrak{g}^*.$$

by

$$J(q,p)(\xi) = p(\sigma_q(\xi))$$

where we write

$$\sigma_q : \mathfrak{g} \to T_qQ$$

for the infinitesimal action of the Lie algebra: namely:

$$\sigma_q(\xi) = \frac{d}{dt}|_{t=0} exp(t\xi)q$$

For those familiar with Abraham-Marsden-Ratiu notation, they write $\xi_Q(q)$ for $\sigma_q(\xi)$. This map σ_q is a linear map for each $q \in Q$. Taking duals we have

$$\sigma_q^* : T_q^*Q \to \mathfrak{g}^*.$$

We can rewrite the formula for the momentum map then according to:

Proposition 3. *The momentum map for the G action on T^*Q is the map J : $T^*Q \to \mathfrak{g}^*$ given by $J(q,p) = \sigma_q^*(p)$.*

Now, the tangent space at q to the G-orbit through q is $im(\sigma_q)$. It recall the basic linear algebraic fact that the kernel of the transpose (or dual) of a linear map L equals the orthogonal complement (or annihilator) of its image that :

$$J(q,p) = 0 \iff p \perp G \text{ orbit through } q.$$

Let us use the given Riemannian metric on Q to identify TQ and T^*Q. (This identification is the Legendre transformation associated to any Lagrangian $K + U$, K the kinetic energy for the given metric.) Let us also fix an inner product on $\} -$ the Killing metric if that choice is available. Then both T_qQ and $\mathfrak{g}$ are inner product spaces. The transpose of the infinitesimal generator σ_q becomes the tangent version of the momentum map:

$$J(q,v) = \sigma_q^t(v).$$

Moreover, by the linear algebra remark above:

$$ker(d\pi(q))^\perp = \{v \in T_qQ : J(q,v) = 0\}.$$

Now observe that by restriction we have a linear isomorphism $d\pi_q : ker(d\pi(q))^\perp \to T_{\pi(q)}S$ and that the shape space metric on S was defined by declaring this isomorphism to be an isometry. We have shown: *The metric on those phase space points for which $J = 0$ coincides, under projection, with the shape space metric.*

Orthogonal to those velocities v with $J(q,v) = 0$ we have the group directions, consisting of velocities of the form $v = \sigma_q(\xi)$ for some $\xi \in \mathfrak{g}$. The kinetic energy of such velocities is given by:

$$\langle \sigma_q(\xi), \sigma_q(\xi) \rangle_q = \langle \xi, \sigma_q^t \sigma_q(\xi) \rangle_{\mathfrak{g}}.$$

Definition 2. *The moment of inertia tensor at $q \in Q$ is the symmetric non-negative form $\sigma_q^t \sigma_q$ on $\mathfrak{g}$.*

So, we have pointwise linear isomorphisms $T_q Q \cong T_s S \oplus \mathfrak{g}$ under which the metric becomes:

$$ds_Q^2(q) \cong ds_S^2 \oplus I_q.$$

Said in terms of subspaces of $T_q Q$ this splitting isomorphism is the *horizontal-vertical splitting:*

$$T_q Q = ker(\sigma_q^t) \oplus Im(\sigma_q$$

with consequent orthogonal projections $T_q Q \to ker(\sigma_q^t), Im(\sigma_q)$.

Exercise 4. *Verify that the two terms of $v = (v - \sigma_q A_q v) + \sigma_q A_q v$ with*

$$A_q = I_q^{-1} \sigma_q^t : T_q Q \to \mathfrak{g}$$

implement the orthogonal projections of the horizontal-vertical splitting. Use this splitting to prove:

Theorem 1. *The kinetic energy is given by*

$$K(q, v) = \frac{1}{2}\{\langle d\pi_q(v), d\pi_q(v) \rangle_S + \langle J, I(q)^{-1} J \rangle_{\mathfrak{g}}\}$$

where $J = J(q, v)$ is the value of the momentum map on (q, v)

and where we have assumed that $I(q)$ is invertible

Exercise 5. *In the case of the planar N-body problem, with $G = S^1$ and so $\mathfrak{g} = \mathbb{R}$ we have that the inertial tensor is the moment of inertia as defined earlier:* $I = \Sigma m_i |q_i|^2 = \frac{1}{m}\Sigma m_i m_j r_{ij}^2$.

Combining this exercise with theorem 1 establishes formula 1.

2.1. **Principal bundles.** An action is called "locally free" if the kernel of σ_q is zero for all q. This asserts that the isotropy groups are everywhere discrete.

Exercise 6. *An action is locally free if and only if I_q is everywhere positive definite, and hence invertible.*

Now, free implies locally free. Let us suppose that the G action on Q is free. Then $Q \to S = Q/G$ is a principal G-bundle, and the spaces $ker(d\pi_q)^{\perp}$ are horizontal spaces for a connection on this bundle. We call this connection the "natural connection" or the "mechanical connection".

Exercise 7. *The connection one-form $A \in \Omega^1(Q, \mathfrak{g})$ for the natural mechanical connection is given by $A_q(v) = I_q^{-1} J(q, v), v \in T_q Q$.*

This language allows us to relate mechanical concepts such as angular momentum, and moment of inertia to the gauge theoretic concepts of a connection, its horizontal space, and its connection one-form.

2.2. **Sundman identity 1.** The generalization to the N-body problem in 2-dimensions is

$$K = \frac{1}{2}\{\dot{r}^2 + r^2 K_{shape} + r^{-2} J\}$$

This splitting is sometimes called the Saari decomposition. Don Saari observed that by splitting velocities into three parts: homothety ($\dot{r}$), rotatiaon, and what is left, we get this decomposition. Some algebra now yields:

$$2IK - (r\dot{r})^2 - |J|^2 = r^4 K_{shape}$$

The inequality

$$2IK - (r\dot{r})^2 - |J|^2 \geq 0$$

is called *Sundman's inequality.*

Exercise 8. *In the N-body case in 3-dimensional space, we have the Legendre transformation* $p_a = m_a \dot{q}_a$, $\dot{q}_a \in \mathbb{R}^3$, *The angular momentum is then* $J = \Sigma m_a q_a \times \dot{q}_a$. *Compute, using basic vector identities, that the tangent vector* $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_N)$ *is orthogonal (rel. to the mass metric) to all rotations through* q *if and only if* $J(q, \dot{q}) = 0$.

Recall symplectic reduction at 0. As a manifold, this reduced space is the subquotient $J^{-1}(0)/G$ of T^*Q. As a symplectic manifold, its symplectic form is obtained by noting that $J^{-1}(0)$ is a co-isotropic submanifold of the symplectic manifold (T^*Q, ω_Q), and that the G-orbits within this submanifold are the integral leaves of the kernel of ω_Q restricted to $J^{-1}(0)$.

Proposition 4. *The symplectic reduced space of* T^*Q *at* 0 *by the free lifted action of* G *is* T^*S *where* $S = Q/G$ *is the shape space for* Q.

TIME PERMITTING: What the reduced space looks like a non-zero values of the momentum.

Exercise 9. . *In the case of* $Q = \mathbb{R}^{dN}$, *the N-body configurartion space, with its mass metric, and for* $G = SO(d)$ *acting 'diagonally' the momentum map is given* *by*

$$J(q,p) = \Sigma q_a \wedge p_a \in \Lambda^2 \mathbb{R}^d = so(d)^*$$

.

3. Metric and shape metric. Planar three-body problem. Explicit formulae via Jacobi vectors.

After going to center of mass frame we have $Q = \mathbb{C}^2$. Variables which diagonalize the moment of inertia are the Jacobi vectors. [DESCRIBE]

Let us think of $\mathbb{C}^2 = Cone(S^3(1))$. The rotation group acts on the $S^3(1)$ part. Forming the metric quotient we find:

$$\mathbb{C}^2/S^1 = Cone(S^2(1/2))$$

since

$$S^3(1)/S^1 = S^2(1/2)$$

Write r for the cone generator variable – the distance from triple collision. We have $r^2 = I$. Let ϕ, θ be coordinates on the two-sphere.

Then

$$K = \frac{1}{2}(\dot{r})^2 + r^2(\dot{\phi}^2 + \cos^2\phi\dot{\theta}^2) + \frac{1}{r^2}J^2$$

while

$$U = \frac{1}{r}\tilde{U}(\phi,\theta).$$

If we now assume a Riemannian metric on Q, and use it to identify TQ with T^*Q we have that $J(q,p) = 0$ if and only if ...

Associated to any foliation of a manifold Q we have, in the cotangent bundle of Q, the space of all covectors which annihilate vectors tangent to the leaves. Let us denote this space by $H^*Q \subset T^*Q$. It is a co-isotropic submanifold with kernel at (q,p) the set of vectors $(v,0)$ with v tangent to the leaf through q. Consequently, if the leaf space $Q/\mathcal{F}$ is a nice manifold, then the "reduced space" H^*Q/ker is canonically isomorphic to the cotangent bundle of the leaf space S.

In case of Riemannian submersion, the fibers are the leaves of a foliation and S itself is the leaf space.

4. Sundman estimates ? HERE? .

5. McGehee blow-up

Triple collision is an essential singularity for the three-body problem. This singularity can be blown up to a sphere by rescaling variables. If we slow down time on approach to collision, so that instead of taking a finite amount of time to reach triple collision, it takes an infinite amount of time, then the resulting equations are analytic all the way to triple collision. The surprising result is a flow on a 7-manifold, which has the remnants of triple collision as an invariant 6-dimensional submanifold called the "collision manifold" on which the flow is gradient-like. The flow on this collision manifold governs near-triple collision dynamics.

This tool has become a central tool in mathematical celestial mechanics. since it was introduced by Dick McGehee in 1973.

Change variables in the planar three-body problem from $(q,v) \in TQ = \mathbb{R}^8$ to

$$r = \sqrt{I(q)}$$

$$s = q/r \in S(Q) = S^3$$

$$z = \sqrt{r}v$$

(The new variables (r, s, z) still move about in an 8-dimensional manifold, z varying in the 4-dimensional space Q.) Change time by When the ODEs are rewritten in these variables, one finds that they contain a common singular factor of $r^{-3/2}$. Consequently if we redefine time to a new parameter τ according to

$$\frac{d}{d\tau} = r^{3/2}\frac{d}{dt}$$

we get a system of ODEs which is analytic all the way down to triple collision. This system is

$$r' = \nu r$$

$$s' = z = \frac{1}{2}\nu s$$

$$z' = \nabla U(s) + \frac{1}{2}\nu z$$

where

$$\nu = \langle z, s\rangle$$

Blow-up replaces triple collision $r = 0$ with a three- sphere. The variety $r = 0$ is an invariant submanifold for the blown-up equations. Topologically, we have added It is called the collision manifold.

equations now are analytic all the way to $r = 0$.

Exercise 10. *Verify that the McGehee substitutions yield the claimed equations.*

Exercise 11. *Generalize the McGehee substitutions and equations to the d-dimensional N-body problem.*

Commentary We said that triple collisions in the three-body problem acts like an essential singularity for the problem. Binary collisions do not. If bodies 1 and 2 are colliding, with 3 a finite distance away, and if we apply a Levi-Civita transformation to the difference vector $q_1 - q_2$, reparameterizing time according to Levi-Civita, and leaving q_3 alone, we compute that the resulting dynamics is analytic *through* the collision. We have analytically continued solutions through binary collision in a way that depends analytically on initial conditions. It is a theorem that such an analytic continuation through triple collision is impossible. (R. Easton; others.)

6. Sundman estimates and theorem

Theorem 2. *(N =3) If a solution has a triple collision then its total angular momentum J is zero.*

$$2IK \geq \|J\|^2.$$

Refined Sundman estimate: Let K_{sh} denote the quadratic form associated to shape space. Then

$$2IK - (r\dot{r})^2 - |J|^2 \geq r^2 K_{sh}$$

by in a manner which also depends analytically on initial conditions. A central problem, solved by Sundman for $N = 3$, was to understand the asymptotics of approach to triple collision. The solution to this problem was made much simpler, and became a powerful tool, in the hands of McGehee. He blew-up triple collision into a sphere, and slowed down time so that it takes forever to reach triple collision. His methods are central to modern approaches in celestial mechanics and are intimately connected to shape space.