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THE STRUCTURE OF REDUCED COTANGENT PHASE SPACES FOR NON-FREE GROUP ACTIONS

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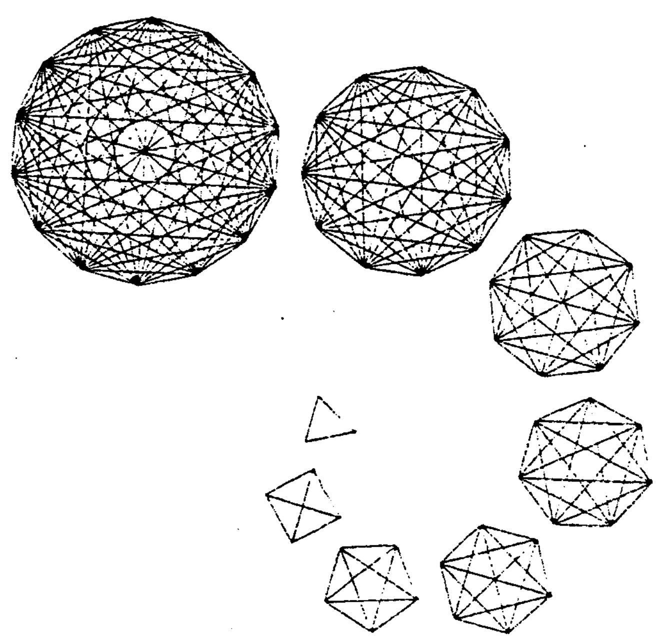

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The Structure of Reduced Cotangent Phase Spaces for Non-Free Group Actions

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ABSTRACT

This paper extends previous results concerning the structure of the reduced phase space for a lifted group action on a cotangent bundle. The main difference between this and earlier papers is that we do not assume that the group action is free. It is shown that if certain regularity conditions and a dimension count hold then the reduced space is itself a cotangent bundle. In general this cotangent bundle does not have the canonical symplectic structure but has an added "magnetic term". Many examples are presented in the concluding section.

1980 Subject Classification: 58F05, 53C99.

Key Words and Phrases: Symplectic manifold, reduced phase space, momentum map.

Introduction

The purpose of this paper is to prove a generalization of the theorems of Smale [1970], Satzer [1977], Marsden (Abraham and Marsden [1978, Theorems 4.3.3 and 4.5.6] and Marsden [1981, Lect. 4]) and Kummer [1981]. These papers show that the reduced space in the sense of Marsden and Weinstein [1974], for the free action of a Lie Group G on Q lifted to T^*Q is embedded as a subbundle of $T^*(Q/G_\mu)$, with equality iff $\mathfrak{g} = \mathfrak{g}_\mu$. Here $\mu \in \mathfrak{g}^*$ is the value at which the reduced manifold is constructed, and G_μ is the isotropy group for the coadjoint action. In this paper we show that a similar result holds in the non-free case with the analog of $\mathfrak{g} = \mathfrak{g}_\mu$ being

$$\dim \mathfrak{g} - \dim \mathfrak{g}_\mu = 2(\dim \mathfrak{g}_Q - \dim \mathfrak{g}_Q^\mu) \qquad \text{(D)}$$

where $\dim \mathfrak{g}_Q$ is the dimension of the isotropy of the G action on Q, and $\dim \mathfrak{g}_Q^\mu$ is the isotropy dimension for the action of G_μ on Q, with these dimensions assumed constant on relevant submanifolds. This situation occurs for Jacobi's "elimination of the node" i.e. the standard action of $SO(3)$ on $T^*(\mathbb{R}^3\backslash\{0\})$. We show that the result applies to $SO(n)$ acting on $T^*\mathbb{R}^n$ and also includes a result of Planchart [1982] concerning the case in which Q is a symmetric space. We shall also discuss the 'magnetic terms' that are studied in Kummer [1981] and their interpretation as the μ-component of the curvature of a connection.

This paper deals only with finite dimensional manifolds, although many of the results are valid in the infinite dimensional case. Probably the right abstract infinite dimensional analog of the finite dimensional statement (D) is that

$$\mathfrak{g}_Q/\mathfrak{g}_Q^{\mu} \hookrightarrow \mathfrak{g}/\mathfrak{g}_{\mu}$$

is a Lagrangian embedding. This statement will be clarified in the appendix.

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§1. Basic Assumptions and Construction

The lifted G action has an Ad^* equivariant momentum map $J: T^*Q \to \mathfrak{g}^*$ given by

$$J(\alpha_q)(\xi) = \alpha_q(\xi_Q(q)) \quad \text{for} \quad \xi \in \mathfrak{g} \tag{J}$$

where ξ_Q is the vector field on Q induced by the action:

$$\xi_Q(q) = \frac{d}{dt}(\exp t\xi)\cdot q|_{t=0} = \sigma_q(\xi)$$

Then the reduced phase space at $\mu \in \mathfrak{g}^*$ is

$$P_\mu = J^{-1}(\mu)/G_\mu$$

Abraham and Marsden [1978] and Kummer [1981] implicitly assume that $\tau(J^{-1}(\mu)) = Q$, where $\tau: T^*Q \to Q$ is the canonical projection, and construct an embedding $P_\mu \to T^*(Q/G_\mu)$. The difference here is that we only assume that $\tau(J^{-1}(\mu))$ is a submanifold of Q, which we will call Q^μ throughout. A smooth map $P_\mu \to T^*(Q^\mu/G_\mu)$ is constructed and under the condition (D) of the introduction this is a diffeomorphism. It reduces to the previous constructions when $Q = Q^\mu$, which occurs when the action on Q is locally free. (see remark at the end of section).

The two-form on $T^*(Q^\mu/G_\mu)$ needed to make the map symplectic is given in formula (5.1), and is seen to contain a 'magnetic term' as in previous papers. As Kummer [1981] does, we interpret (§6) this term as the μ-component of the 'curvature' of a 'connection' on $Q^\mu \to Q^\mu/G_\mu$. This component is a standard two-form on Q^μ/G_μ precisely because μ is G_μ invariant.

To aid intuition in what follows, the reader may wish to occasionally refer to the motivating example (§7), $G = SO(3)$ acting on $Q = \mathbb{R}^3$, which, though very simple, contains many elements of the general theory.

We will make the following

Assumptions

(A1) μ is a weakly regular value for J. That is, $J^{-1}(\mu)$ is a submanifold of T^*Q with $T_\alpha J^{-1}(\mu) = \text{Ker } TJ_\alpha$. $\tau: T^*Q \to Q$ has constant rank when restricted to $J^{-1}(\mu)$ and $Q^\mu = \tau(J^{-1}(\mu))$ is a submanifold of Q. G_μ acts properly on $J^{-1}(\mu)$, hence on Q^μ, and $Q^\mu \to Q^\mu/G_\mu$ is a submersion. In particular Q^μ/G_μ is a manifold. (All manifolds assumed without boundary).

The basic ingredient in constructing the map $P_\mu \to T^*(Q^\mu/G_\mu)$ is, as in previous papers, the assumption

(A2) There is a smooth G_μ equivariant one-form, α_μ, with values in $J^{-1}(\mu)$.

(The difference here is that Q^μ may not be all of Q, so α_μ has domain Q^μ instead of Q.)

Remark

In examples, α_μ is usually not hard to compute (see examples in this paper, in Abraham and Marsden [1978] and Kummer [1981]). One expects, by Marsden [1978, Theorem 4.5.6] and the interpretation of α_μ as the μ-component of a connection (§6) that in reasonable cases, e.g. if $J^{-1}(\mu) \to P_\mu$ is a fibre bundle, α_μ exists.

Outline of Construction

Using α_μ, a G_μ equivariant map $\psi: J^{-1}(\mu) \to \text{Ker } J^\mu$ is constructed in §2. Here J^μ is the Ad^* equivariant momentum map for the G_μ action on T^*Q^μ, defined by formula (J), except Q is replaced by Q^μ, and $\mathfrak{g}$ by $\mathfrak{g}_\mu$.

There is a natural map $f: \mathrm{Ker}\, J^{\mu} \to T^{*}(Q^{\mu}/G_{\mu})$ making $T^{*}(Q^{\mu}/G_{\mu}) \approx (\mathrm{Ker}\, J^{\mu})/G_{\mu}$, that is, f is a submersion with fibres the G_{μ} orbits. For completeness, this is shown in the appendix.

Since ψ is equivariant, we have the commutative diagram

$$\begin{array}{ccc} J^{-1}(\mu) & \xrightarrow{\ \psi\ } & \mathrm{Ker}\, J^{\mu} \\ {\scriptstyle \pi_{\mu}} \downarrow & & \downarrow {\scriptstyle f} \\ P_{\mu} & \xrightarrow{\ \overline{\psi}\ } & T^{*}(Q^{\mu}/G_{\mu}) \end{array}$$

Figure 1

which defines the desired map $\overline{\psi}$, which is continuous by the openness of the projections. As is always the case with equivariant mappings, ψ *is one-to-one or onto precisely when the quotient map* $\overline{\psi}$ *is*.

Remarks Concerning Assumptions

(A1): In practice, it is easier to check that Q^{μ} is a submanifold of Q, than to check the statements in (A1) concerning J, so we consider the alternative assumption

(A1') Q^{μ} *is a submanifold of* Q *satisfying*

$$\dim Q^{\mu} = \dim Q - (\dim \mathfrak{g}_q - \dim \mathfrak{g}_q^{\mu}) \tag{D1}$$

and $Q^{\mu} \to Q^{\mu}/G_{\mu}$ *is a submersion.*

This is useful because of the next:

Lemma. Suppose (A2) and (A3) hold. Then (A1') holds iff (A1) holds.

The proof of this lemma relies on machinery developed over the next three sections, so we relegate its proof to the appendix.

It is important to note, especially for computation, that

$$Q^\mu = \{q \in Q : \mathfrak{g}_q \subseteq \text{Ker } \mu\}$$

This is easy to see: $q \in Q^\mu$ iff $\exists \alpha_q \in J_q^{-1}(\mu)$, that is an $\alpha_q \in T_q^* Q$ satisfying

$$\langle \alpha_q, \sigma_q \xi \rangle = \langle \mu, \xi \rangle \quad \forall \xi \in \mathfrak{g}.$$

On the one hand, if there is such an α_q then $\mathfrak{g}_q = \text{Ker } \sigma_q \subseteq \text{Ker } \mu$. On the other hand, if $\mathfrak{g}_q \subseteq \text{Ker } \mu$ then the above formula defines a linear functional α_q on $T_q G \cdot q$, which we can then extend to all of $T_q Q$ (say by letting it be 0 on a complementary subspace to $T_q G \cdot q$), thus getting an $\alpha_q \in J_q^{-1}(\mu)$.

This simple expression has the immediate consequence that if Q_{free} denotes those elements of Q at which the action is locally free ($\mathfrak{g}_q = 0$) then

$$Q_{\text{free}} \subseteq Q^\mu \quad \text{for all } \mu \in \mathfrak{g}^*.$$

The previous theorems all dealt with the case $Q_{\text{free}} = Q$, i.e. $Q = Q^\mu$.

Note also that if there are any trivial isotropy groups, $\mathfrak{g}_q = 0$, then there is no hope of Q providing a new example, i.e. one not covered by the $Q_{\text{free}} = Q$ cases, for if $Q^\mu \neq Q_{\text{free}}$, then Q must contain q's with $\mathfrak{g}_q \neq 0$, on the other hand, it contains all q's with $\mathfrak{g}_q = 0$, hence the constancy of dimension assumtion (A3) would not hold.

(A2): In examples, α_μ is usually not hard to compute. And in a large class of examples, one can show that (A2) is satisfied as follows. If G_μ is compact and G has a bi-invariant metric (in particular, if G is compact) then (A2) is implied by:

(A2'): There is a G_μ equivariant one form $\tilde{\alpha}_\mu$ on Q^μ with values in $(J^\mu)^{-1}(\mu \restriction \mathfrak{g}_\mu) \subseteq T^*Q$.

If in addition G_μ acts freely on Q^μ, then (A2'), and hence (A2), automatically holds, either by a theorem in Abraham and Marsden [1978, Theorem 4.5.6] or by Kummer's interpretation of $\tilde{\alpha}_\mu$ as the μ-component of a G_μ connection on $Q^\mu \to Q^\mu/G_\mu$.

To extend the $\tilde{\alpha}_\mu$ of (A2') to an α_μ of (A2) put a metric on Q such that G_μ acts by isometries and such that $\sigma_q(\mathfrak{g}_\mu) \perp \sigma_q(\mathfrak{g}_\mu^\perp)$ for $q \in Q^\mu$. Any $v \in T_qQ$ can be written uniquely as $v^T + v^\perp$ $v^T \in T_qQ$ and $v^\perp \perp T_qQ^\mu$. One checks that

$$\alpha_\mu(q)(v) = \tilde{\alpha}_\mu(q)\cdot v^T$$

defines such an extension.

(A3): Throughout this paper G_x will mean the isotropy subgroup of G at $x \in X$ relative to the G action on X, and $G_x^\mu = G_\mu \cap G_x$ will denote the isotropy for the same action restricted to G_μ. The corresponding Lie algebras will be denoted $\mathfrak{g}_x$ and $\mathfrak{g}_x^\mu$. Note that whenever $F: X \to Y$ is a G-equivariant map, then $G_x \subseteq G_{F(x)}$, and that if F is one-to-one, then $G_x = G_{F(x)}$. Using (A3) and (A2) we can prove the important:

Isotropy Lemma.

$$\mathfrak{g}_\alpha = \mathfrak{g}_q^\mu$$

whenever $\alpha \in J^{-1}(\mu)$, where $q = \tau(\alpha)$.

Proof. By the equivariance of J and τ, $G_\alpha \subseteq G_\mu, G_q$. Hence

$$G_\alpha \subseteq G_\mu \cap G_q = G_q^\mu \tag{I}$$

On the other hand, α_μ is G_μ equivariant so $G_q^\mu \subseteq G^\mu_{\alpha_\mu(q)} \subseteq G_q^\mu$. Thus $G_q^\mu = G_{\alpha_\mu(q)}$ and so the result follows from constancy of the dimenisions of G_α. ■

§2. The construction of $\psi : J^{-1}(\mu) \to \mathrm{Ker}\, J^\mu$

If we subtract α_μ, we get a map $\phi : J^{-1}(\mu) \to \mathrm{Ker}_{Q^\mu} J$, where $\mathrm{Ker}_{Q^\mu} J = J^{-1}(0) \cap \tau^{-1}(Q^\mu)$. To be explicit, let

$$\phi(\alpha_q) = \alpha_q - \alpha_\mu(q)$$

Note that ϕ is a fibre-preserving (in fact fibre-affine) diffeomorphism, (its inverse is adding α_μ). It is equivariant because α_μ is.

We then have the diagram in Figure 2.

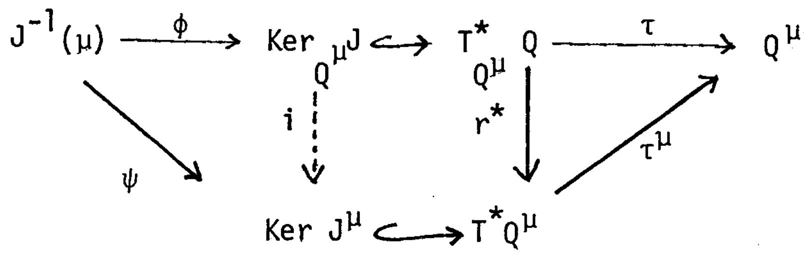


Figure 2

Here $r : Q \hookrightarrow Q^\mu$ is the inclusion. The dotted line means that we are going to show that $i = r^* | \mathrm{Ker}_{Q^\mu} J$ actually maps into $\mathrm{Ker}\, J^\mu$. This will be done in the proof of Theorem 1. Assuming this, we see that r is G_μ equivariant (Q^μ is G_μ invariant) hence r^*, i and finally $\psi = i \circ \phi$ are equivariant.

We are interested in when $\bar{\psi}$, the induced base map, is a diffeomorphism, for which we'll want to know when ψ is. Since ϕ is a diffeomorphism, the only obstruction to $\psi = i \circ \phi$ being a diffeomorphism is i, a vector bundle map over Q^μ. This reduces the diffeomorphism questions to linear algebra and dimension counting, as expressed in:

Theorem 1. Let (A1), (A2) and (A3) hold and i be as just defined. Then i

(i) maps into $\text{Ker } J^\mu$

(ii) is onto iff for all $q \in Q^\mu$, $T_q G_\mu \cdot q = T_q Q^\mu \cap T_q(G \cdot q)$

(iii) is one-to-one iff for all $q \in Q^\mu$, $T_q G \cdot q + T_q Q^\mu = T_q Q$

Also (iv) The dimension count (D1) of §1 holds.

(v) i is a diffeomorphism iff condition (D) in the introduction holds.

We shall prove this in §4 below.

§3. The Main Result

Theorem 2. Let (A1), (A2) and (A3) hold. Then $\bar{\psi}: P_\mu \to T^* Q^\mu / G_\mu$ is a continuous map onto a subbundle of $T^*(Q^\mu/G_\mu)$. (ii) and (iii) of Theorem 1 hold with $\bar{\psi}$ in place of i, and (v) also holds if 'diffeomorphism' is replaced by 'homeomorphism'.

Assume $J^{-1}(\mu) \to P_\mu$ is a fibre bundle, and condition (D) holds. Then $\bar{\psi}$ is a diffeomorphism.

Remark. If the isotropy subgroups, G_α, for the G_μ action on $J^{-1}(\mu)$ are all conjugate (in G_μ) $J^{-1}(\mu) \to P_\mu$ is automatically a fibre bundle (with fibre G_μ/G_α).

Proof of Theorem 2: As mentioned immediately below Figure 1, $\bar{\psi}$ is one-to-one or onto exactly when ψ is, so, according to the discussion immediately preceding the statement of Theorem 1, exactly when i is. If ψ is a homeomorphism so is $\bar{\psi}$, by openness of the projections in Figure 1.

To see that im $\overline{\psi}$ is a subbundle, note ϕ^{-1} of Figure 2, and π_μ of Figure 1, are onto, and $\overline{\psi}\circ\pi_\mu\circ\phi^{-1} = f\circ i$. So im $\overline{\psi}$ = im$f\circ i$ and the latter is a subbundle because f and i are constant rank vector bundle morphisms.

If $J^{-1}(\mu) \to P_\mu$ is a fibre bundle, one can use local sections to show $\overline{\psi}$ is smooth. If (D) holds $(\psi, \overline{\psi})$ is then a smooth bundle homeomorphism, so Ker $J^\mu \to T^*(Q^\mu(G_\mu)$ is also fibre bundle and one can now use local sections of this bundle to show $\overline{\psi}^{-1}$ is smooth. ■

§4. Proof of Theorem 1

(i) - (iii) follow from a lemma from linear algebra. Set

$$T_qQ = V,\ T_qG\cdot q = V_1,\ T_qQ^\mu = V_2 \text{ and } T_qG_\mu\cdot q = V_3$$

for $q \in Q^\mu$. Note $V_3 \subseteq V_1 \cap V_2$ (since Q^μ is G_μ-invariant). For $U \subseteq W$ subspaces of V, the W annihilator of U will be denoted $\text{ann}_W\, U = \{\alpha \in W^*: \alpha(U) = 0\}$.

Then Ker $J_q = \text{ann}_V V_1$, Ker $J_q^\mu = \text{ann}_{V_2} V_3$, and fibre-wise i is the composition

$$\text{ann}_V V_1 \hookrightarrow V^* \xrightarrow{\text{restrict}} V_2^*$$

Lemma. This map, which we will also call i, maps onto $\text{Ann}_{V_2} V_1 \cap V_2 \subset \text{ann}_{V_2} V_3$. Hence it is onto $\text{ann}_{V_2} V_3$ iff $V_3 = V_1 \cap V_2$. It is one-to-one iff $V_1 + V_2 = V$.

Proof. First note that if α annihilates V_1, then $\alpha \upharpoonright V_2$ annihilates $V_1 \cap V_2$. Hence i maps into $\text{ann}_{V_2}(V_1 \cap V_2)$. To see it is onto, find a subspace W such that $V_1 \subseteq W$, $V_1 \cap V_2 = W \cap V_2$ and $W + V_2 = V$. (This is easy to do in the finite dimensional case.) Then, for $\alpha \in \text{ann}_{V_2} V_1 \cap V_2$, set $\beta(v_2 + w) = \alpha(v_2)$. This is well defined, for if $v_2 + w = v_2' + w'$ then $v_2 - v_2' = w - w' \in V_2 \cap W = V_1 \cap V_2$. Hence $\alpha(v_2 - v_2') = 0$ or $\alpha(v_2) = \alpha(v_2')$. Note β annihilates W, hence V_1, and $i\beta = \beta \upharpoonright V_2 = \alpha$.

To prove the remark concerning one-to-oneness, note that for $\alpha \in \text{ann}_V V_1$, $i\alpha = \alpha \upharpoonright V_2 = 0$ iff α annihilates $V_1 + V_2$. That is Ker $i = 0$ iff $V_1 + V_2 = V$

Making the substitutions above we see that this lemma is precisely (i), (ii), (iii).

(iv) - (v): The following notation will be used: if S is a submanifold of T^*Q, $\alpha \in S$, then we set

$$S_q = S \cap T_q^*Q$$

$$\text{and} \quad S_{Q^\mu} = S \cap \tau^{-1}(Q_\mu).$$

Also, let $T_\alpha^v S$ = vertical tangent space to S at $\alpha = \text{Ker } T(\tau \upharpoonright S)_\alpha = T_\alpha S \cap T_\alpha^v(T^*Q)$ where $\tau : T^*Q \to Q$ as usual, and let

$$\tau^\mu = \tau | J^{-1}(\mu)$$

and $\tau^0 = \tau | J^{-1}(0)$.

For the various dimensions we will use

$$g = \dim \mathfrak{g},\ g_q = \dim \mathfrak{g}_q,\ g_q^\mu = \dim \mathfrak{g}_q^\mu, \text{ and } g_\alpha = \dim \mathfrak{g}_\alpha$$

Note that $g_\alpha = g_q^\mu$ from the isotropy lemma. We will need the following facts:

Facts

(a) $T\tau_\alpha^\mu(T_\alpha J^{-1}(\mu)) = T_q Q^\mu$ (independent of α)

(b) $\dim J^{-1}(\mu) = 2n - (g - g_\alpha)$.

(c) $T_\alpha^v J^{-1}(\mu) \simeq T_{\phi(\alpha)}^v \operatorname{Ker} J \simeq \operatorname{Ker} J_q$ and have dimension $n - (g - g_q)$.

Proofs. (a) Since $\tau^\mu : J^{-1}(\mu) \to Q$, we know

$$T\tau_\alpha^\mu(T_\alpha J^{-1}(\mu)) \subseteq T_q Q^\mu \qquad \text{(I)}$$

Now $\alpha_\mu : Q^\mu \to J^{-1}(\mu)$ satisfies $\tau^\mu \circ \alpha_\mu = 1_{Q^\mu}$, so $T\alpha_{\mu_q} Q^\mu \subseteq T_{\alpha_\mu(q)} J^{-1}(\mu)$ and $T\tau_{\alpha_\mu(q)}^\mu \circ T\alpha_{\mu_q}(T_q Q^\mu) = T_q Q^\mu$, from which it follows that (a) is satisfied for $\alpha = \alpha_\mu(q)$. For general α the result follows from that for $\alpha_\mu(q)$ using the assumption of constant rank of τ, the fact that $T\tau | TJ^{-1}(\mu) = T\tau^\mu$, and the inclusion (I).

(b) For any momentum map J, on a symplectic manifold (P,ω), TJ_α is the transpose of the composition

$$\xi \mapsto \sigma_\alpha(\xi) \mapsto \omega(\sigma_\alpha(\xi), \cdot)$$

where $\sigma_\alpha : \mathfrak{g} \to T_\alpha P$ is the map induced by the G action. This follows directly from the definitions. Thus

$$\text{Ker } TJ_\alpha = \sigma_\alpha^\perp$$

$\perp$ in the sense of the symplectic form, ω, so dim Ker TJ_α = $\dim(T_\alpha(T^*Q))$ - dim im σ_α = $2n - (g-g_\alpha)$.

(c) $\phi: J^{-1}(\mu) \to \text{Ker}_{Q^\mu} J$ is a fibre preserving diffeomorphism, hence induces the first isomorphism.

The second is a result of the fact that $\text{Ker}_{Q^\mu} J$ is a vector sub-bundle of $T^*_{Q^\mu}Q$. This is seen by considering J as a vector bundle morphism of vector bundles over Q^μ

$$J: T^*_{Q^\mu}Q \longrightarrow Q^\mu \times \mathfrak{g}^*$$

It has constant rank $g-g_q$ since im $J_q = \mathfrak{g}_q^\perp$ ($\perp$ is the duality sense) as is clear from the definition (J). Hence $\text{Ker}_{Q^\mu} J$ is a vector subbundle of fibre dimension dim Ker J_q = dim $T^*_q Q$ - dim im J_q = $n - (g-g_q)$. Finally for any vector bundle, a fibre is canonically isomorphic to a vertical tangent space, at any point in that fibre.

(iv). Putting these results together we have:

$$\begin{aligned} \dim Q^\mu &= \dim T_q Q^\mu \\ &= \dim \text{im } T\tau_\alpha^\mu \quad \text{(by (a))} \\ &= \dim T_\alpha J^{-1}(\mu) - \dim \text{Ker } T\tau_\alpha^\mu \\ &= 2n - (g - g_\alpha) - (n - (g - g_q) \quad \text{(by (b) and (c))} \\ &= n - (g_q - g_\alpha) \end{aligned}$$

as desired.

(v) Since i is a vector bundle map over Q^μ, it is a diffeomorphism iff it is one-to-one and onto. From the Remark concerning isotropy (§1) $g_\alpha = g_q^\mu \equiv \dim G_\mu \cap G_q$, in this case.

Recall $T_q G_\mu \cdot q \subseteq T_q G \cdot q \cap T_q Q^\mu$, always, since Q^μ is G_μ invariant. So (ii) translates to:

"i is onto iff $g_\mu - g_q^\mu = \dim T_q G\cdot q \cap T_q Q^\mu$."

Similarly (iii) may be written:

"i is one-to-one iff $[g-g_q] + [n - (g_q - g_\alpha)] - \dim T_q G\cdot q \cap T_q Q^\mu = n$."

Putting these results together we get

"i is one-to-one and onto iff $(g-g_q) + (n-(g_q-g_\alpha)) - (g_\mu - g_q^\mu) = n$"

after some algebra and using $g_\alpha = g_q^\mu$ this becomes

$$g - g_\mu = 2(g_q - g_q^\mu)$$

which is (D).

§5. The Symplectic Structure on $T^*(Q^\mu/G_\mu)$

A form on $T^*(Q^\mu/G_\mu)$ is now found which makes our map $\overline{\psi}$ symplectic.

We have the following commutative diagram, with canonical one-forms written next to their cotangent bundles where throughout this section all vector bundles except $T^*(Q^\mu/G_\mu)$ are considered as vector bundles over Q^μ.

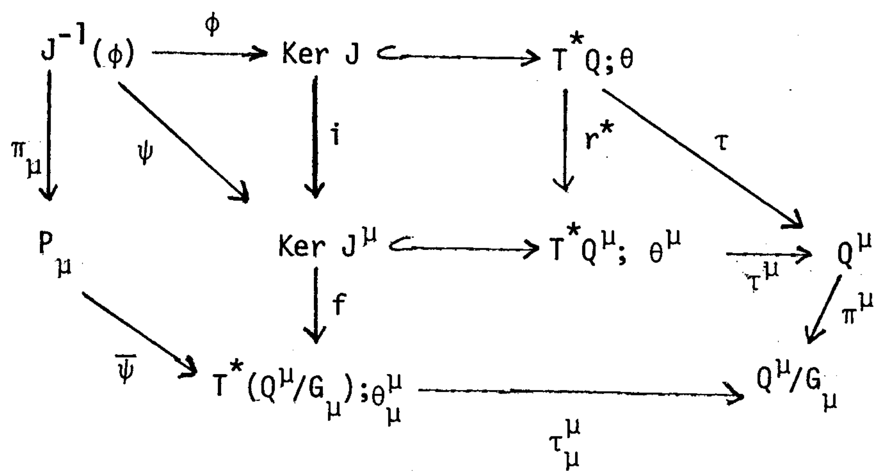


Figure 3

The following notation will be used: Let $N \subseteq M$ be manifolds and γ a form on either N or M with values in the exterior algebra bundle over M. e.g. if γ is a one-form, we mean either a section of $T^*_N M$ or T^*M. Then $\gamma \restriction N$ will mean $j^*\gamma : N \to T^*N$ where j is the inclusion $N \hookrightarrow M$.

Theorem 3. Let α_μ be as in (A2).

(a) There is a unique two-form $\overline{d\alpha_\mu}$ on Q^μ/G_μ with $\pi^{\mu *}\overline{d\alpha_\mu} = d\alpha_\mu \restriction Q^\mu$.

(b) Set:

$$\omega^\mu_\mu = \omega^\mu_0 - \tau^{\mu *}_\mu \overline{d\alpha_\mu} \tag{5.1}$$

where $\omega_0^\mu = -d\theta_\mu^\mu$ is the canonical two-form on $T^*(Q^\mu/G_\mu)$. Let ω_μ denote the canonical symplectic form on P_μ. Then $\overline{\psi}:(P_\mu,\omega_\mu) \to (T^*Q^\mu/G_\mu, \omega_\mu^\mu)$ is symplectic.

Remarks. $\tau_\mu^{\mu*}\overline{d\alpha_\mu}$ is known as a magnetic term.

Note that by $d\alpha_\mu$ we mean the standard two-form $d(\alpha^\mu \restriction Q^\mu)$ on Q^μ.

Proof of Theorem 3. (b): We will assume (a). It will be proved below. We wish to show $\overline{\psi}^*\omega_\mu^\mu = \omega_\mu$. Recall ω_μ is defined by $\pi_\mu^*\omega_\mu = \omega \restriction J^{-1}(\mu)$ where $\omega = -d\theta$ is the canonical symplectic form on T^*Q. Since π_μ is a submersion, π_μ^* is injective on forms. Also $f\circ\psi = \overline{\psi}\circ\pi_\mu$. So it is equivalent to show:

$$\psi^* f^* \omega_\mu^\mu = \omega \restriction J^{-1}(\mu) \tag{5.2}$$

We will show

(i) $f^*\theta_\mu^\mu = \theta^\mu \restriction \mathrm{Ker}\ J^\mu$

(ii) $i^*(\theta^\mu \restriction \mathrm{Ker}\ J^\mu) = \theta \restriction \mathrm{Ker}\ J$

(iii) $\phi^*(\theta \restriction \mathrm{Ker}\ J) = \theta - \tau_\mu^*\alpha_\mu \restriction J^{-1}(\mu)$

From this it follows that

$$\psi^* f^* \theta_\mu^\mu = \phi^* i^* f^* \theta_\mu^\mu = \theta - \tau_\mu^*\alpha_\mu \restriction J^{-1}(\mu)$$

Hence

$$\psi^* f^* \omega_0^\mu = -d(\psi^* f^* \theta_\mu^\mu) = \omega + \tau_\mu^* d\alpha_\mu | J^{-1}(\mu) \tag{5.3}$$

Now $\tau^{\mu}_{\mu} \circ f = \pi^{\mu} \circ \tau^{\mu} | \text{Ker } J^{\mu}$, so

$$f^* \tau^{\mu *}_{\mu} \overline{d\alpha_{\mu}} = \tau^{\mu *} \pi^{\mu *} \overline{d\alpha_{\mu}}$$
$$= \tau^{\mu *} d\alpha_{\mu} \quad \text{(from (a))}$$

Since $\tau^{\mu} \circ \psi = \tau | J^{-1}(\mu)$ we get

$$\psi^* f^* \tau^{\mu}_{\mu} \overline{d\alpha_{\mu}} = \psi^* \tau^{\mu *} d\alpha_{\mu} = \tau^* d\alpha_{\mu} | J^{-1}(\mu) .$$

Subtracting this from (5.3) gives (5.2), the desired result.

Proofs of (i) - (iii).

It is shown in the appendix that f may be defined by

$$\langle f(\alpha), \pi_* v \rangle = \langle \alpha, v \rangle$$

where $\pi = \pi^{\mu}$, and $\langle \, , \, \rangle$ denotes the vector-covector pairing on the appropriate space. In the following X denotes a vector tangent to the appropriate space:

(i): $$(f^* \theta^{\mu}_{\mu})(\alpha)(X) = \theta^{\mu}_{\mu}(f(\alpha))(f_* X)$$
$$= \langle f(\alpha), \tau^{\mu}_{\mu *} f_* X \rangle$$
$$= \langle f(\alpha), (\pi_{\mu *} \tau^{\mu}_* X) \rangle \quad (\tau^{\mu}_{\mu} \circ f = \pi_{\mu} \circ \tau^{\mu})$$

$$= \langle \alpha, \tau^{\mu}_* X \rangle$$
$$= \theta^{\mu}(\alpha)(X).$$

(ii): $(i^*\theta^\mu)(\alpha)(X) = \theta^\mu(i\alpha)(i_*X)$

$$= \langle r^*\alpha,\ \tau^\mu_* i_* X\rangle$$

$$= \langle r^*\alpha\ \tau_* X\rangle \quad (\text{since } \tau^\mu \circ i = \tau)$$

$$= \langle \alpha,\ r_*\tau_* X\rangle$$

$$= \langle \alpha,\ \tau_* X\rangle \quad (\text{since } r\circ\tau = \tau \text{ when restricted to } \operatorname{Ker}_{Q^\mu} J)$$

$$= \theta(\alpha)(X).$$

(iii): $\phi^*\theta_0(\alpha)(X) = \theta_0(\phi\alpha)\phi_*X)$

$$= \langle \phi\alpha,\ \tau_*\phi_* X\rangle$$

$$= \langle \phi\alpha\ \tau_* X\rangle \quad (\text{since } \tau\circ\phi = \tau \text{ when restricted to } J^{-1}(\mu))$$

$$= \langle \alpha - \alpha_\mu, \tau_* X\rangle$$

$$= \langle \alpha,\ \tau_* X\rangle - \langle \alpha_\mu,\ \tau_* X\rangle$$

$$= \theta(\alpha)(X) - \tau^*\alpha_\mu(X)$$

(a) We will use [q] to denote $\pi^\mu(q) = G_\mu$ orbit through q, and π to denote $\pi^\mu{}_*$. Since $\pi^{\mu *}$ is injective on forms, uniqueness is clear.

<u>Existence of $\overline{d\alpha_\mu}$:</u> Set

$$\overline{d\alpha_\mu}([q])(\pi_q X,\ \pi_q Y) = d\alpha_\mu(X,Y) \tag{5.4}$$

We need only check that this is well defined, for clearly then $\pi^{\mu *}\overline{d\alpha_\mu} = d\alpha_\mu$. Since π_q is onto for each q, $\overline{d\alpha_\mu}$ is a form on Q^μ/G_μ as long as our definition is:

(i) Independent of π_q, in the sense that if $\pi_q X = \pi_q X'$ and $\pi_q Y = \pi_q Y'$ then $d\alpha_\mu(q)(X,Y) = d\alpha_\mu(q)(X',Y')$.

(ii) Independent of q, in the sense that if $q' = gq$, $g \in G_\mu$ $\pi_{q'} X' = \pi_q X$, and $\pi_{q'} Y' = \pi_q Y$ then $d\alpha_\mu(q)(X,Y) = d\alpha_\mu(gq)(X',Y')$.

These are precisely the statements that, in the terminology of Kobayashi-Nomizu [1963], $d\alpha_\mu$ is a 'tensorial form'. That is,

(i') $$d\alpha_\mu(q)(X,Y) = 0$$

If either $\pi_q X$ or $\pi_q Y$ are 0.

(ii') $$g^{-1*} d\alpha_\mu = d\alpha_\mu$$

To see that (i') implies (i), suppose, X,X'; Y,Y' are as in the statement of (i) and that (i') holds. Then, since $\pi_q(X-X') = 0$ we have $d\alpha_\mu(q)(X-X',Y) = 0$ or $d\alpha_\mu(q)(X,Y) = d\alpha_\mu(X',Y)$. Likewise $d\alpha_\mu(q)(X',Y) = d\alpha_\mu(X',Y'')$.

Then (ii') together with (i) imply (ii). For we have

$$\begin{aligned}(g^{-1*} d\alpha_\mu)(gq)(X',Y') &= d\alpha_\mu(q)(g_*^{-1}X', g^{-1}{}_*Y') \text{ (by definition)} \\ &= d\alpha_\mu(gq)(X',Y') \text{ (by (ii'))}\end{aligned}$$

and $\pi_q g_*^{-1} X = \pi_q X$, $\pi_q g_*^{-1} Y = \pi_q Y$ since $\pi^\mu \circ g^{-1} = \pi^\mu$, hence applying (i), we're done.

(ii') Is merely a restatement of α_μ's equivariance.

(i') By skew symmetry we need only consider the case $\pi_q X = 0$. Any such X lies in $T_q G_\mu \cdot q$ hence can be written $\xi_Q(q)$ for some $\xi \in \mathfrak{g}_\mu$. Suppose Y were a vector field in the vicinity of q. Then

$$d\alpha_\mu(q)(\xi_Q,Y) = L_{\xi_Q}(\alpha_\mu(Y)) - L_Y(\alpha_\mu(\xi_Q)) - \alpha_\mu[\xi_Q,Y]$$

$$= \frac{d}{dt}\,\alpha_\mu(\exp t\xi\cdot q)(Y_{\exp t\xi q}) - L_Y(\mu(\xi)) - \alpha_\mu[\xi_Q,Y]$$

(the middle term occurs because α_μ maps into $J^{-1}(\mu)$)

$$= \frac{d}{dt}\,(\exp - t\xi)^*\alpha_\mu(q)(Y_{\exp t\xi}) - \alpha_\mu[\xi_Q,Y]$$

(by α_μ's equivariance)

$$= \alpha_\mu(q)\,\frac{d}{dt}\,(\exp\cdot t\xi)_*\,Y_{\exp - t\xi}) - \alpha_\mu(q)[\xi_Q,Y]$$

$$= \alpha_\mu(q)[\xi_Q,Y] - \alpha_\mu(q)[\xi_Q,Y]$$

$$= 0\ . \quad \blacksquare$$

§6. Curvature Interpretation of the Magnetic Term

The relationship between the magnetic term in (5.1) and curvature for the bundle $\pi^\mu: Q^\mu \to Q^\mu/G_\mu$ is essentialy as in Kummer [1981]. An interesting example of the realization of α_μ as a "μ-connection" is provided in the $SL(2,\mathbb{C})$ example of the next section. But all the examples presented in this paper including this one, are trivial in the sense that $d\alpha_\mu = 0$. Although no nontrivial examples of the theory in this section, besides those which can be worked out with the old theory, have been found yet, this section is included for completeness and also in hopes that others may discover examples.

We will start with a workable extension of the definition of connection. Recall $\sigma_q: \mathfrak{g} \to T_qQ$ denotes the linear map

$$\sigma_q(\xi) = \xi_Q(q)$$

and that Ker $\sigma_q = \mathfrak{g}_q$.

Definition. A μ-connection on Q is a smooth family of linear maps

$$\Gamma_q : T_qQ \to \mathfrak{g}/\mathfrak{g}_q, \quad q \in Q^\mu$$

satisfying

$$\begin{aligned} &\text{(a)} \quad g^*(\Gamma_{gq}) = \mathrm{Ad}_g \circ \Gamma_q, \quad g \in G_\mu \\ &\text{(b)} \quad \Gamma_q \circ \sigma_q \text{ is the projection } p_q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_q. \end{aligned} \tag{6.1}$$

Remarks concerning this definition:

If G acts freely then, as mentioned in §1, $Q^\mu = Q$. Also $\mathfrak{g}_q = \{0\}$ and this definition reduces to the standard one for a connection.

Part (a): Here Ad_g is interpreted as the isomorphism $\mathfrak{g}/\mathfrak{g}_q \to \mathfrak{g}/\mathfrak{g}_{gq}$ induced by the fact that $\mathrm{Ad}_g \mathfrak{g}_q = \mathfrak{g}_{gq}$, so both sides of (a) are maps $T_qQ \to \mathfrak{g}/\mathfrak{g}_{gq}$. Note that this formula gives the proper transformation law for a connection on a principal left G_μ bundle.

Part (b): translates to

$$\mathrm{Ker}\, \Gamma_q \oplus \mathrm{im}\, \sigma_q = T_qQ \qquad (q \in Q^\mu) \tag{6.1.b'}$$

To see that $\mathrm{Ker}\, \Gamma_q \cap \mathrm{im}\, \sigma_q = \{0\}$, say $v = \sigma_q \xi \in \mathrm{im}\, \sigma_q$ and $\Gamma_q \sigma_q \xi = 0$. Then $\xi \in \mathfrak{g}_q = \mathrm{Ker}\, \sigma_q$, so $v = 0$. To see that $\mathrm{Ker}\, \Gamma_q + \mathrm{im}\, \sigma_q = T_qQ$, let $v \in T_qQ$ and write $v = (v - \sigma_q\xi) + \sigma_q\xi$ where ξ represents the coset $\Gamma_q v$, that is $p_q\xi = \Gamma_q v$. Then $\Gamma_q(v - \sigma_q\xi) = \Gamma_q v - p_q\xi = 0$.

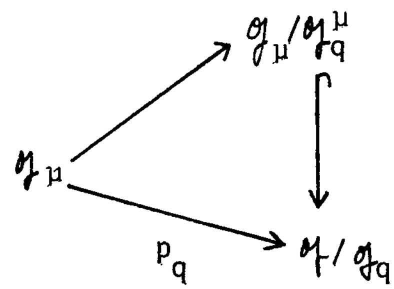
Set

$$\Gamma_q^\mu = \Gamma_q | T_qQ^\mu, \quad \sigma_q^\mu = \sigma_q | \mathfrak{g}_\mu$$

then the remark for part (b) implies that $\mathrm{Ker}\,\Gamma^{\mu}_{q} + (\mathrm{im}\,\sigma_q \cap T_qQ^{\mu}) = T_qQ^{\mu}$. From now on assume condition (D) holds. Then, by (ii) of Theorem 1, $\mathrm{im}\,\sigma_q \cap T_qQ^{\mu} = \mathrm{im}\,\sigma^{\mu}_{q}$, so that:

$$\mathrm{Ker}\,\Gamma^{\mu}_{q} \oplus \mathrm{im}\,\sigma^{\mu}_{q} = T_qQ^{\mu} \tag{6.2}$$

Note that $\Gamma^{\mu}_{q}\circ\sigma^{\mu}_{q} = p_q : \mathfrak{g}_{\mu} \to \mathfrak{g}/\mathfrak{g}_q$ factors G_{μ} equivariantly:



so, in the particular case that G_{μ} acts freely on Q^{μ} (so $\mathfrak{g}^{\mu}_{q} = 0$), Γ^{μ}_{q} is a standard connection on the principal bundle $\pi^{\mu}: Q^{\mu} \to Q^{\mu}/G_{\mu}$.

Because of (6.2) one can now define a horizontal projection

$$h_q : T_qQ^{\mu} \to \mathrm{Ker}\,\Gamma^{\mu}_{q}$$

and hence curvature

$$R_q = h^{*}_{q}d\Gamma^{\mu}_{q} = h^{*}_{q}d\Gamma \restriction T_qQ^{\mu}$$

Both h and R satisfy most of the standard formulas, in particular, the Ad equivariance formulas, for standard connections.

Also, from (6.2) we see $\pi^{\mu}_{*} \restriction \mathrm{Ker}\,\Gamma^{\mu}_{q}$ is an isomorphism onto $T_{[q]}Q^{\mu}/G_{\mu}$ because π^{μ}_{*} is a submersion and $\mathrm{Ker}\,\pi^{\mu}_{*} = \mathrm{im}\,\sigma^{\mu}_{q}$. So we have the notion of the vertical lift, $X^{*}_{q} \in \mathrm{Ker}\,\Gamma^{\mu}_{q}$, for a vector $X \in T_{[q]}Q^{\mu}/G_{\mu}$. It satisfies $\pi^{\mu}_{*}X^{*}_{q} = X$.

We can now think of R as a two-form, Ω, on Q^μ/G_μ, with values in the 'associated bundle' Q^μ/G_μ, with values in the 'associated bundle' $Q^\mu \times_{Ad} \mathfrak{g}/\mathfrak{g}_q$, in the standard way

$$\Omega_{[q]}(X,Y) = [q, R_q(X_q^*, Y_q^*)]_{G_\mu}$$

or equivalently (6.3)

$$\Omega_{[q]}(\pi_*^\mu X_q, \pi_*^\mu Y_q) = [q, R_q(X_q, Y_q)]_{G_\mu}$$

where G_μ-equivalence relation is $(g,v) \sim (gq, Ad_g v)$.

Recall (remark at end of §1).

$$\mathfrak{g}_q \subseteq \text{Ker } \mu \quad \text{if} \quad q \in Q^\mu \tag{6.4}$$

So one can consider μ as a linear functional on $\mathfrak{g}/\mathfrak{g}_q$ which we will denote by μ_q:

$$\langle \mu_q, p_q\xi \rangle = \langle \mu, \xi \rangle \tag{6.5}$$

Because μ is G_μ-invariant we have

$$Ad_g^* \mu_{gq} = \mu_q \tag{6.6}$$

and more importantly, <u>the</u> <u>μ-component of</u> Ω

$$(\Omega_\mu)_q = \mu_q \circ \Omega_q$$

<u>makes sense as a standard two-form on</u> Q^μ/G_μ.

<u>Theorem 4.</u> $\alpha_\mu(q) = \Gamma_q^* \mu_q$ (6.7)

<u>with $*$ in the linear algebra sense, defines a one-form of the type needed in the basic construction, that is, α_μ satisfies (A2) of §1.</u>

<u>Also</u>

$$\overline{d\alpha_\mu} = \Omega_\mu \tag{6.8}$$

<u>where $\overline{d\alpha_\mu}$ has the same meaning as in Theorem 3.</u>

<u>Proof</u>. First we check α_μ satisfies (A2). $\alpha_\mu(q) \in J^{-1}(\mu)$:

$$\begin{aligned}\langle J(\Gamma_q^*\mu_q),\xi\rangle &= \langle \Gamma_q^*\mu_q, \sigma_q\xi\rangle \quad \text{(by (J))}\\ &= \langle \mu_q, \Gamma_q\sigma_q\xi\rangle\\ &= \langle \mu_q, P_q\xi\rangle \quad \text{(by (6.1.b))}\\ &= \langle \mu,\xi\rangle \quad \text{(by (6.5))}\end{aligned}$$

α_μ is equivariant:

$$\begin{aligned}g\cdot(\alpha_\mu(q)) = g^{-1*}(\Gamma_q^*\mu_q) = \mu_q\circ g^{-1*}\Gamma_q \quad &(\text{since } \Gamma_q^*\mu_q = \mu_q\circ\Gamma_q)\\ = \mu_q\circ \mathrm{Ad}_{g^{-1}}\circ\Gamma_{gq} \quad &\text{(by (6.1.a))}\\ = \mathrm{Ad}_{g^{-1}}^*\mu_q\circ\Gamma_{gq}\quad&\\ = \mu_{gq}\circ\Gamma_{gq} \quad &\text{(by (6.6))}\\ = \alpha_\mu(gq)\quad&\end{aligned}$$

To show (6.8), note that from (6.3) and the G_μ invariance of μ, that $\pi_q^*\Omega_\mu = \mu_q\circ R_q$. So according to Theorem 3, we need only show

$$d\alpha_\mu \upharpoonright Q^\mu = \mu_q \circ R$$

Since both sides are 'tensorial form', that is zero on vertical vectors (see (5.4)) we need only check this on horizontal vectors, i.e., for vector fields X, Y in Ker Γ^μ. Then:

$$\begin{aligned} d\alpha_\mu(q)(X,Y) &= d(\mu_q \circ \Gamma_q)(X,Y) \\ &= X(\mu_q(\Gamma_q(Y)) - Y(\mu_q(\Gamma_q(X)) - \mu_q \circ \Gamma_q[X,Y] \\ &= -\mu_q \circ \Gamma_q[X,Y] \end{aligned}$$

and

$$\begin{aligned} \mu_q \circ R(X,Y) &= \mu_q h^* d\Gamma_q^\mu(X,Y) = \mu_q d\Gamma_q^\mu(X,Y) \\ &= \mu_q\{X(\Gamma_q^\mu Y) - Y(\Gamma_q^\mu X) - \Gamma_q^\mu[X,Y]\} \\ &= -\mu_q \cdot \Gamma_q^\mu[X,Y] \end{aligned}$$

and these are equal. Note it is essential that the splitting (6.2) is that induced by (6.1.b').

Remark. Combining Kummer's [1981] interpretation of the cohomology class of Ω_μ as the obstruction to being able to find a symplectomorphism

$$(P_\mu, \omega_\mu) \to (T^*(Q^\mu/G_\mu),\ d\theta_\mu^\mu)$$

in the case $Q^\mu = Q$, $G_\mu = G$ with $Q \to Q/G$ a principal bundle and Duistermaat and Heckman's [1982] method of comparing these different cohomology classes for different μ in case G is a torus, along with some type of reduction of G's action and momentum map to those of G's maximal torus in the case

G compact, it seems that it should be possible to come up with an obstructional interpretation for our Ω_μ at least for compact G.

§7. Examples

Vector Space Examples

These concern the case $Q = \mathbb{R}^n$ or $\mathbb{C}^n$ with the canonical inner product, and $G = SO(n)$ or $SU(n)$. Then $T^*Q \cong Q \times Q$ via the inner product, $\xi_Q(q) = \xi \cdot q$, the lifted action becomes the diagonal action $g(q,p) = (gq,gp)$ and the momentum map is

$$J(q,p)(\xi) = \langle p, \xi \cdot q \rangle$$

We can associate $\mathfrak{g}$ with $\mathfrak{g}^*$ via some constant multiple of its Killing form, $\langle \mu, \xi \rangle = c\,\mathrm{tr}\,\mu\xi^*$ where the $*$ means real transpose or complex adjoint. Under this association the coadjoint action becomes the adjoint action and is an action of isometries.

The statement "$J(q,p) = \mu$" becomes "for all $\xi \in \mathfrak{g}$ $\langle p, \xi \cdot q \rangle = \langle \mu, \xi \rangle$". Now $\langle p, \xi \cdot q \rangle \doteq \Sigma p_i \overline{\xi_{ij} q_j} = \Sigma p_i \overline{q_j} \overline{\xi}_{ij} = \mathrm{tr}(pq)\xi^*$, where $(pq)_{ij} = p_i \overline{q_j}$. Letting $\mathrm{Pr}: \mathfrak{gl}(n) \to \mathfrak{g}$ denote orthogonal projection onto $\mathfrak{g}$, we may finally write this as "for all $\xi \in \mathfrak{g}$ $\mathrm{tr}\,\mathrm{Pr}(pq)\xi^* = c\,\mathrm{tr}\,\mu\xi^*$." So by the nondegeneracy of the Killing form:

$$J(q,p)_{ij} = \mu_{ij} = \frac{1}{c} \Pr(pq)_{ij} = \frac{1}{2c} \begin{cases} p_i q_j - q_i p_j & \text{for } SO(n) \\ p_i \bar{q}_j - q_i \overline{p_j} - \frac{2\sqrt{-1}}{n} \operatorname{Im} \langle p,q \rangle \delta_{ij}, & \text{for } SU(n) \end{cases} \tag{E1}$$

$SO(3) = G$ on $\mathbb{R}^3 = Q$.

As Lie algebras, so(3) is isomorphic to $\mathbb{R}^3$ with the cross product. By adjusting the constant c of (E1), we have $J(q,p) = q \times p$, the familiar angular momentum.

Suppose $\mu \neq 0$, $\mu \in \mathfrak{g}^* \simeq \mathbb{R}^3$. Then

$$J^{-1}(\mu) = \{(q,p): q \times p = \mu\} \subseteq \{(q,p): q,p \in \mu^\perp - \{0\}\}$$

(note from the formula $\|\mu\| = \|q\|\|p\| \sin\theta$ that $J_q^{-1}(\mu)$ is a line in $\mu^\perp \subseteq \mathbb{R}^3$).

We claim that

$$Q^\mu = \mu^\perp - \{0\}$$

That $Q^\mu \subseteq \mu^\perp - \{0\}$ is clear. To see the other inclusion, define $\alpha_\mu : \mu^\perp - \{0\} \to J^{-1}(\mu)$ by $\alpha_\mu(q) = (q, \beta(q))$ where $\beta(q)$ is the vector in $\mathbb{R}^3$ uniquely determined by the condition that $\{q, \beta(q), \mu\}$ forms a right handed orthogonal basis for $\mathbb{R}^3$ with $q \times \beta(q) = \mu$. Thus $\tau(J^{-1}(\mu)) = Q^\mu \subseteq \mu^\perp - \{0\}$.

In fact α_μ is

$$G_\mu = SO(\mu^\perp) \simeq SO(2)$$

equivariant, so satisfies (A2).

$$G_q^\mu = \{I\}$$

since the identity is the only element of SO(3) which fixes two linearly independent vectors, namely μ and q. Since $G_\alpha \subseteq G_q^\mu$, $\alpha = (q,p) \in J^{-1}(\mu)$, we have

$$G_\alpha = G_q^\mu = \{I\}$$

Also

$$G_q = SO(q^\perp)$$

so condition (D1) is

$$\dim Q^\mu = 2 = 3 - (1-0) = \dim Q - (\dim \mathfrak{g}_q - \dim \mathfrak{g}_\alpha)$$

and

$$\pi^\mu : Q^\mu \to Q^\mu/G_\mu \simeq \mathbb{R}^2\backslash\{0\}/SO(2) \simeq \text{ ray}$$

is a principal (circle) bundle. Thus, according to Lemma 2, §1 (A1) holds also.

Condition (D) holds:

$$\dim \mathfrak{g} - \dim \mathfrak{g}_\mu = 3 - 1 = 2(1-0) = 2(\dim \mathfrak{g}_Q - \dim \mathfrak{g}_Q^\mu)$$

So, according to Theorem 2, we should have a diffeomorphism $\overline{\psi}:P_\mu \to T^*(Q^\mu/G_\mu)$

$$\overline{\psi}:P_\mu \to T^*(Q^\mu/G_\mu) \simeq T^*(\text{ray})$$

$\overline{\psi}$ will be constructed following the prescription in §1 and §2.

$\phi:J^{-1}(\mu) \to \text{Ker } J$ is given by $(q,p) \to (q,p - \beta(q))$ and $\text{im } \phi = \text{Ker } J^\mu$, so the map i of $\psi = i\circ\phi$ is unnecessary with our identifications.

The projection $f:\text{Ker } J^\mu \to T^*(Q^\mu/G_\mu)$ is $f(q,\gamma) = (\|q\|, \langle q,\gamma\rangle)$ upon making the identification for $T^*(Q^\mu/G_\mu)$ above. So, for $[q,p] \in P_\mu$

$$\bar{\psi}([q,p]) = (\|q\|, \langle q,p - \beta(q)\rangle) = (\|q\|, \langle q,p\rangle).$$

This is easily checked to be a diffeomorphism, directly.

We will show $d\alpha_\mu = 0$, hence according to theorem 3, $\bar{\psi}$ is a symplecto-morphism with T^*Q^μ/G_μ having its standard structure.

Let e_1, e_2 be an orthonormal basis for μ such that $[e_1,e_2,\mu]$ is right handed, i.e. $\mu = \|\mu\| e_1 \times e_2$. This induces coordinates on Q^μ in which

$$\alpha_\mu(x_1,x_2) = \frac{\|\mu\|}{\|q\|^2} (-x_2e_1 + x_1e_2)$$

where $q = x_1e_1 + x_2e_2$. In differential form notation $\alpha_\mu = \|\mu\|(-x_2dx_1 + x_1dx_2)/(x_1^2 + x_2^2)$. It is well known that this form is closed.

<u>SO(n) on $\mathbb{R}^n$</u>

One finds that if $\mu = J(q,p) \neq 0$ then μ is orthogonally similar to a matrix of the form

$$\left[\begin{array}{cc|c} 0 & \theta & \\ -\theta & 0 & 0 \\ \hline 0 & & 0 \end{array}\right] \qquad \theta \neq 0$$

Assuming μ of this form, p and q then lie in the x-y plane and the rest of the example proceeds just as for $n = 3$ with the result that the reduced spaces are cotangent bundles of rays.

$G = SU(3)$ on $Q = \mathbb{C}^3$

For $\mu \neq 0$ regular, one finds Q^{μ} is a three-dimensional cone minus origin in a $\mathbb{C}^2 \subseteq \mathbb{C}^3$ and that G_{μ} is the two-torus acting on this $\mathbb{C}^2$. The behavior of Q^{μ} as μ moves from one Weyl chamber to another is rather interesting. In any case Q^{μ}/G_{μ} is a ray, so the reduced space is again the cotangent bundle of a ray.

$SU(n)$ on $\mathbb{C}^n$

The generalization from SU(3) to SU(n) is essentially the same as from SO(3) to SO(n). Again the reduced spaces for acceptable μ are cotangent bundles of rays.

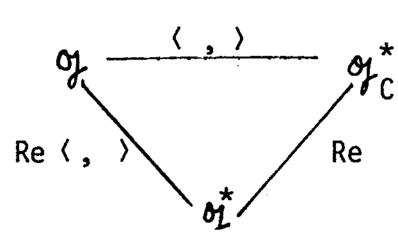
$SL(2,\mathbb{C})$ on its Lie Algebra

The action is the adjoint action. The infinitesimal generators are $\sigma_q(\xi) = [\xi,q]$. It is well known, or easily checked, that $[\xi,q] = 0$ iff $\xi = zq$ for some $z \in \mathbb{C}$, when $q \neq 0$, hence $\mathfrak{g}_q$ is the complex span of q.

Recall (remark at end of §1) that $Q^{\mu} = \{q : \mathfrak{g}_q \subseteq \operatorname{Ker} \mu\}$. Now using the complex Killing form

$$\langle \eta, \xi \rangle = \operatorname{tr} \eta \xi$$

we have a natural complex isomorphism $\mathfrak{g} \approx \mathfrak{g}_{C}^{*}$, the complex-linear functionals on $\mathfrak{g}$. By taking real parts we get the following commuting diagram of isomorphisms:



where $\mathfrak{g}^*$ is the <u>real</u> dual. Suppose $\mu \in \mathfrak{g}^*$, $\mu \neq 0$ and let $\hat{\mu}$ be the corresponding element in $\mathfrak{g}$. Then $q \in Q^\mu$ <u>iff</u> $\mathrm{Re}\langle \hat{\mu}, zq \rangle = 0$ for all $z \in \mathbb{C}$, which in turn is true iff $\langle \hat{\mu}, q \rangle = 0$. Dropping our hats, we see

$$Q^\mu = \{q \in Q : \langle q, \mu \rangle = 0 \text{ and } q \neq 0\} \tag{1}$$

(the $q \neq 0$, because $\mathfrak{g}_0 = \mathfrak{g} \not\subset \ker \mu$).

In particular, for

$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Q^μ consists of the $q = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$, $(\alpha,\beta) \neq (0,0)$.

$$G_\mu = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in C^* \right\} \simeq C^*$$

whre $\mathbb{C}^*$ means $C \backslash \{0\}$.

$$z \cdot q = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & \beta z^2 \\ \alpha z^{-2} & 0 \end{pmatrix} \tag{2}$$

$$G_q = \{\pm I\}$$

The dimension counts are

$$[g - g_\mu] = 6 - 2 = 2(2-0) = 2[g_q - g_q^\mu]$$

for (D1), and

$$\dim Q^{\mu} = n - [g_q - g_q^{\mu}] = 6-[2-0] = 4$$

for (D2), so we have a winner.

The equivalence class of an (α,β) in $C^2\backslash\{0\} \simeq Q^{\mu}$ under the G action can be described

$$[\alpha,\beta] = \{(\alpha z^{-1},\beta z):z \in C^*\}$$

Note that the determinant function,

$$f(\alpha,\beta) = \det\begin{pmatrix} & \beta \\ \alpha & \end{pmatrix} = -\alpha\beta$$

is constant on equivalence classes. Also, if $\alpha\beta \neq 0$ and $\alpha'\beta' = \alpha\beta$ then $(\alpha',\beta') = (\alpha(\alpha'/\alpha), \beta(\alpha/\alpha'))$ hence $[\alpha,\beta] = [\alpha',\beta']$. This is easily checked to define a diffeomorphism

$$Q^{\mu}/G_{\mu} \tilde{\to} C\backslash\{0\}$$

<u>if we begin by taking</u> Q <u>to be the G-invariant subset of</u> $s\ell(2,C)$ <u>with non-zero determinant</u>. Thus, by the main result:

$$P_{\mu} \simeq T^*(C\backslash\{0\})$$

<u>Remark</u>. If we do not make the restriction $\det \neq 0$, the quotient space Q^{μ}/G_{μ} is the <u>non-Hausdorff</u> manifold, "$\mathbb{C}$", consisting of $\mathbb{C}$ except with <u>two origins</u> (corresponding to the equivalence classes [1,0] and [0,1]) at which the topology becomes non-Hausdorff. The main result $P_{\mu} \simeq T^*$"$\mathbb{C}$", still holds, with the non-Hausdorff cotangent bundle interpreted in the obvious way.

The symplectic structure is the standard one if we take the equivariant one-form to be:

$$\alpha_\mu(q) = \frac{1}{2}\begin{pmatrix} 0 & -\beta^{-1} \\ \beta^{-1} & 0 \end{pmatrix}$$

where $q = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$. Equivariance is easily checked. The fact that $\alpha_\mu(q) \in J^{-1}(\mu)$ follows from the facts that $\langle \alpha_\mu(q), [\xi,q] \rangle = \langle [q,\alpha_\mu(q)],\xi \rangle$ and that $[q,\alpha_\mu(q)] = \mu$. Note that as a complex valued one-form on Q^μ

$$\alpha_\mu(q)\begin{pmatrix} & \beta' \\ \alpha' & \end{pmatrix} = \frac{1}{2}\,\mathrm{tr}\begin{pmatrix} 0 & -\alpha^{-1} \\ \beta^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta' \\ \alpha' & 0 \end{pmatrix} = \frac{1}{2}[-\alpha^{-1}\alpha' + \beta^{-1}\beta']$$

$$= \frac{1}{2}[-\alpha^{-1}d\alpha + \beta^{-1}d\beta](\alpha',\beta')$$

This is holomorphic, so that, considered as a <u>real</u> one-form

$$d\alpha_\mu = \frac{1}{2}\, d\,\mathrm{Re}(-\alpha^{-1}d\alpha + \beta^{-1}d\beta) = \frac{1}{2}\,\mathrm{Re}\, d(-\alpha^{-1}d\alpha + \beta^{-1}d\beta)$$

$$= \frac{1}{2}\,\mathrm{Re}\, d(\alpha^{-2}d\alpha \wedge d\alpha - \beta^{-2}d\beta \wedge d\beta)$$

$$= 0$$

demonstrating that the structure is standard.

There is a connection interpretation for α_μ. Since $\sigma_q(\xi) = [\xi,q]$, a connection, Γ, would be an equivariant family of maps

$$v = [\xi,q] + \lambda_q \longmapsto \xi + \mathfrak{g}_q = \Gamma_q(v) \in \mathfrak{g}/\mathfrak{g}_q$$

where we are using the facts that $\mathfrak{g}_q$ is the complex span of q and $\mathfrak{g} = \{[\xi,q] : \xi \in \mathfrak{g}\} \oplus \mathfrak{g}_q$. One finds, that in fact

$$v = [\frac{1}{4 \det q} [q,v],q] + (-\mathrm{tr}(vq)/2 \det q)q$$

(This is just linear algebra, made easier by writing $q = \alpha X + \beta Y$, $v = v_X X + v_y Y + v_H H$ where X,Y,H are the standard basis for $s\ell(2,\mathbb{C})$, and uding their commutation relations). So

$$\Gamma_q(v) = \frac{1}{4 \det q} [q,v].$$

Γ is equivariant:

$$z^*\Gamma_{zq}(v) = \Gamma_{z\cdot q}(z\cdot v) = \frac{1}{4 \det q} [\mathrm{Ad}_z q, \mathrm{Ad}_z v]$$

$$= \frac{1}{4 \det q} \mathrm{Ad}_z[q,v] = z\cdot\Gamma_q(v)$$

so is in fact a μ-connection (see (6.1)). A simple calculation shows that α_μ is $\mu\circ\Gamma$:

$$\mu\circ\Gamma_q(v) = \frac{1}{4 \det q} \langle \mu, [q,v] \rangle = \langle \frac{1}{4 \det q} [\mu,q],v \rangle$$

and

$$\frac{1}{4 \det q} [\mu,q] = \alpha_\mu(q)$$

Homogeneous Spaces

Let H be a closed subgroup of G. Then G acts by left translation on the homogeneous space $Q = G/H$ of right cosets. Planchart [1982], and A.S. Mishchenko [1982] show that in case Q is a symmetric space that the reduced space is zero-dimensional. Planchart's method relies on the fact that $\alpha_\mu : Q^\mu \to J^{-1}(\mu)$ is a G_μ equivariant diffeomorphism, so in this sense uses a special case of the methods of this paper. Planchart's work was crucial in the formulation of this paper in that it offered the first (and so far only) computable non-vector space example and also the first example for which G_μ did not act trivially on Q^μ.

We will use the notation $|g|$ for the right coset gH. One computes

$$\xi_{G/H}(|g|) = \pi_* R_{g*}\xi, \ \xi \in g$$

where $\pi : G \to G/H$. So $J(\alpha_{|g|}) = \mu$ is equivalent to

$$\langle \alpha_{|g|}, \pi_* R_{g*}\xi \rangle = \langle \mu, \xi \rangle \quad \forall \xi \in g \tag{1}$$

Since any vector in $T_{|g|}G/H$ can be written $\pi_* R_{g*}\xi$, $\xi \in g$, this defines a 1-form, α_μ, on $Q^\mu \subseteq G/H$ and Q^μ consists of the $|g|$ for which this equation really does define a 1-form. That is, $|g| \in Q^\mu$ iff whenever $\pi_* R_{g*}\xi = 0$ we have $\langle \mu, \xi \rangle = 0$. Now $\pi_{g*} R_{g*}\xi = 0 \iff R_{g*}\xi \in L_{g*}\mathfrak{h}$, where $\mathfrak{h}$ is H's Lie algebra, $\iff \xi \in \mathrm{Ad}_g \mathfrak{h}$. So

$$Q^\mu = \{|g| : \mathrm{Ad}_g \mathfrak{h} \subseteq \mathrm{Ker}\ \mu\}$$

and from the above discussion

$$J_q^{-1}(\mu) = T_q^* G/H \cap J^{-1}(\mu) = \{\alpha_\mu(q)\} \quad \text{for} \quad q \in Q^\mu. \tag{2}$$

Hence $\alpha_\mu = \tau \restriction J^{-1}(\mu)$ and $J^{-1}(\mu)$ and Q^μ are homeomorphic. In fact α_μ is G_μ-equivariant. This can be checked directly, or, more quickly, it follows from the 0-dimensionality of $J_q^{-1}(\mu)$: we know $J^{-1}(\mu)$ is G_μ-invariant and that if $\alpha \in J_q^{-1}(\mu)$ then $g \cdot \alpha \in J_{gq}^{-1}(\mu)$ since both sets are singletons, this means $g \cdot \alpha_\mu(q) = \alpha_\mu(g \cdot q)$. So $\alpha_\mu : Q^\mu \to J^{-1}(\mu)$ is an equivariant homeomorphism, thus induces a homeomorphism

$$Q^\mu / G_\mu \simeq P$$

Assuming (A1) holds, we see that α_μ is an equivariant diffeomorphism and the diffeomorphism $P_\mu \Rightarrow Q^\mu/G_\mu$ induced by $\tau \restriction J^{-1}(\mu) = \alpha_\mu^{-1}$ is precisely the $\overline{\psi}$ of the main result, after identifying Q^μ/G_μ with $\tau^* Q^\mu / G_\mu$'s 0-section:

$$\bar{\psi}([\alpha]) = f(\alpha - \alpha_\mu(q)) = f(0_q) = 0_{[q]} \quad .$$

here $\alpha \in J_q^{-1}(\mu)$ and the second equality occurs because $\alpha = \alpha_\mu(q)$.

Note in particular that $\bar{\psi}: P_\mu \simeq Q^\mu/G_\mu \to T^*(Q^\mu/G_\mu)$ is a homeomorphism iff Q^μ/G_μ is zero-dimensional.

Planchart shows that μ is a weakly regular value of J in the same manner that we do in the proof of our lemma 2. That is, he shows that if Q^μ is a submanifold, then so is $J^{-1}(\mu)$, as in the first part of our proof there (the constancy of $\dim \mathfrak{g}_q$) is automatic here, since $G_{|g|} = gHg^{-1}$). Then he goes through the dimension count (D1) in this special case to show that μ is in fact weakly regular, as we do in the second part of our proof. To show that Q^μ is in fact a manifold takes some work, and we will not go into this.

Condition (iii) of our Theorem 1 automatically holds here, since

$$T_q G \cdot q = T_q Q$$

If condition (ii) of Theorem 1 also holds, we know $\bar{\psi}$ is a homeomorphism by Theorem 2, hence Q^μ/G_μ is zero-dimensional by a previous remark. In this case (ii) is

$$T_q Q^\mu = T_q G_\mu \cdot q \tag{3}$$

which is directly checkable in the symmetric case. We use Planchart's argument.

Proof of (3). In the symmetric case, we have the Cartan decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is H's lie algebra, $\mathfrak{m}$ may be identified with T_0G/H, $0 = eH$, via π_*, and $[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}$, Without loss of generality, we may assume $0 = eH \in Q^\mu$, i.e. $\mathfrak{h} \subseteq \text{Ker}\ \mu$, and do our work at $q = 0$, since for any other $q = gH$, the Cartan decomposition just gets shifted by Adg.

The inclusion $T_q G_\mu \cdot q \subseteq T_q Q^\mu$ is automatic so say $\bar{v} = \pi_* v \in T_q Q^\mu$ and let it be represented by the curve $|c(t)| = \pi(c(t))$ in Q^μ, where $c(t)$ is a curve through e in G representing $v = \frac{d}{dt} c(t)|_{t=0} \in m$. Then $\text{Ad}_{c(t)}\mathfrak{h} \subseteq \text{Ker}\ \mu$, so $\frac{d}{dt}\text{Ad}_{c(t)}\mathfrak{h} = [v,\mathfrak{h}] \subseteq \text{Ker}\ \mu$. Also $[v,\mathfrak{m}] \subseteq \mathfrak{h}$, since $v \in \mathfrak{m}$. Now $\mathfrak{h} \subseteq \ker \mu$, so $[v,\mathfrak{g}] \subseteq \text{Ker}\ \mu$. Therefore $v \in \mathfrak{g}_\mu$ and $\bar{v} = \pi_* v \in T_q G_\mu \cdot q$.

Another Line of Investigation:

is in connection with some work of Wolf [1975] in which he showed that for μ a regular nilpotent element in the Lie algebra of a semisimple G that its adjoint orbit is diffeomorphic to an open subset of $T^*(G/P)$ where $P \subseteq G$ corresponds to a real poalrization for μ. In particular for $G = SO(n,1)$ one gets $G/P \simeq S^n$ and the orbit is diffeomorphic to $T^*S^n\backslash\{0\text{-section}\}$. It is not clear how our work would be extended to this case, for im $\bar{\psi}$ is inherently a subbundle of $T^*(Q^\mu/G_\mu)$.

run-in? This looks odd.

Appendix 1. The construction of $f:\text{Ker } J^\mu \to T^*Q^\mu/G_\mu$

f is defined by the duality pairing

$$\langle f(\alpha_q), \pi_{*q}v\rangle = \langle \alpha_q, v\rangle$$

where $v \in T_qQ^\mu$ and $\pi = \pi^\mu$ denotes the projection $Q^\mu \to Q^\mu/G_\mu$, which we assume to be a submersion.

We will show f's fibres are G_μ orbits and that f also is a submersion.

First, f is well defined: if $\pi_{*q}v_1 = \pi_{*q}v_2$ then $v_1 - v_2 \in \text{Ker } \pi_{*q} = T_qG_\mu \cdot q$, hence $\langle \alpha_q, v_1 - v_2\rangle = 0$, since $\alpha_q \in \text{Ker } J^\mu$.

To see that the fibres of f are exactly the G_μ orbits we must show: $f(\alpha_q) = f(\alpha_{q'}) \Longleftrightarrow \exists g \in G_\mu \; g^*\alpha_q = \alpha_{q'}$

$\Longleftarrow$: If $g^*\alpha_q = \alpha_{q'}$, then

$$\langle f(\alpha_{q'}), \pi_{*q'}v\rangle = \underbrace{\langle \alpha_{q'}, v\rangle = \langle g^*\alpha_q, v\rangle} = \langle \alpha_q, g_*v\rangle = \langle f(\alpha_q), \pi_{*q}g_*v\rangle$$

$$= \langle f(\alpha_q), \pi_{*q'}v\rangle ,$$

the last equality because $\pi \circ g = \pi$ and $gq' = q$. So $f(\alpha_{q'}) = f(\alpha_q)$.

$\Rightarrow$: Conversely, if $f(\alpha_q) = f(\alpha_{q'})$, then both are forms over the same base point in Q^μ/G_μ, i.e. $\exists g \in G_\mu$, $gq' = q$. We can then turn this string of equalities inside out, that is, the two outside terms are now equal, and we can work inward, meeting at the bracketed terms which tells us $\alpha_{q'} = g^*\alpha_q$.

To see that f is a submersion, note we could also define f by $\pi_q^*f(\alpha_q) = \alpha_q$, or $f(\alpha_q) = \pi_q^{*-1}\alpha_q$. Here $\pi_q^*:T^*_{\pi q}Q^\mu/G_\mu \to T^*_qQ^\mu$ is injective (since π_* is onto) and $\text{im } \pi_q^* = \text{Ker } J_q^\mu$ (this is essentially why f is well defined), so taking the inverse of π_q^* makes sense and $\pi_q^{*-1} = f_q$ is a linear isomorphism $\text{Ker } J_q^\mu \to T^*_{\pi q}Q^\mu/G_\mu$.

Let (η, v), $(\gamma, \pi(v))$ be local trivilization charts for the vector bundles $\mathrm{Ker}\, J^{\mu}$, $T^{*}Q^{\mu}/G_{\mu}$, respectively. So

$$\eta(q,v) = (q,\bar{\eta}(q)v), \quad \text{for } \ker J^{\mu}$$

$$\gamma(\pi q,v) = (\pi q, \bar{\gamma}(\pi q)v), \quad \text{for } T^{*}Q^{\mu}/G_{\mu}$$

where $v \in \mathrm{IR}^{k}$, k = fibre dim $\mathrm{Ker}\, J^{\mu}$ = fibre dim $T^{*}Q^{\mu}$ (= dim $Q^{\mu} - (g_{\mu} - g_{q}^{\mu})$ in the notation of Theorem 1), $\bar{\eta}(q) \in \mathrm{Aut}(\mathrm{IR}^{k}, \mathrm{Ker}\, J_{q}^{\mu})$, and $\bar{\gamma}(q) \in \mathrm{Aut}(\mathrm{IR}^{k}, T_{\pi q}^{*}Q^{\mu}/G_{\mu})$. Then f is $\gamma^{-1} \circ f \circ \eta$ in these coordinates:

$$\gamma^{-1} f \eta(q,v) = (\pi q, \underbrace{(\bar{\gamma}(\pi q)^{-1} \circ f_{q} \circ \bar{\eta}(q))}_{B_q} v)$$

$B_q \in G\ell(K)$ and $q \to \Gamma_q$ is a smooth map $U \to G\ell(K)$. Then in these coordinates, Tf is

$$\left[\begin{array}{c|c} T\pi_q & \partial_q B_q \\ \hline 0 & B_q \end{array}\right]$$

which is onto, since B_q and $T\pi_q$ are.

2. A demonstration that the natural embedding

$$j : \mathfrak{g}_q/\mathfrak{g}_q^\mu \hookrightarrow \mathfrak{g}/\mathfrak{g}_\mu$$

is isotropic for $q \in Q^\mu$, that is, $\operatorname{im} j \subseteq (\operatorname{im} j)^\perp$ where $\perp$ is taken with respect to the canonical symplectic form

$$\omega([\xi],[\gamma]) = -\mu([\xi,\gamma]) = \frac{d}{dt} \mathrm{Ad}^*_{\exp-t\xi}\, \mu(\gamma)\Big|_{t=0}$$

on $\mathfrak{g}/\mathfrak{g}_\mu$ (the brackets inside ω denote cosets).

Recall that $\mathfrak{g}_q \subseteq \operatorname{Ker} \mu$ (remark, end of §1). Then, if $\xi, \gamma \in \mathfrak{g}_q$, $\mathrm{Ad}_{\exp-t\xi}\gamma \in \mathfrak{g}_q \subseteq \operatorname{Ker} \mu$, so

$$\omega(j(\xi + \mathfrak{g}_q^\mu),\ j(\gamma + \mathfrak{g}_q^\mu)) = \frac{d}{dt}\, \mu(\mathrm{Ad}_{\exp-t\xi}\gamma)\Big|_{t=0} = 0$$

In particular, since $\dim \operatorname{im} j + \dim \operatorname{im} j^\perp = \dim \mathfrak{g}/\mathfrak{g}_\mu$ for the finite dimensional case, then j is a Lagrangian embedding iff the dimension count (D) holds.

3. Proof of Lemma, §1

Assume (A2) and (A3).

Statement (iv) of Theorem 1 is that (A1) implies (A1').

For the other implication, assume (A1'). We need to show

(1) $J^{-1}(\mu)$ is a submanifold of T^*Q,

and (2) $T_\alpha J^{-1}(\mu) = \operatorname{Ker} TJ_\alpha$ for $\alpha \in J^{-1}(\mu)$.

In the proof of Thereom 1 (§4, (iv) - (v), fact (c)) we proved

$\mathrm{Ker}_{Q^\mu} J$ is a vector subbundle of $T^*_{Q^\mu} Q$ with fibre dimension $n-(g-g_q)$. This proof is valid in the present situation. Adding α_μ, that is applying the diffeomorphism ϕ^{-1} considered as a map $T^*_{Q^\mu} Q \to T^*_{Q^\mu} Q$, one sees that

$$J^{-1}(\mu) = \phi^{-1}(\mathrm{Ker}_{Q^\mu} J)$$

is a submanifold of $T^*_{Q^\mu} Q$, hence of T^*Q. $\mathrm{Ker}_{Q^\mu} J$ is a vector bundle over Q^μ with projection $\tau | \mathrm{Ker}_{Q^\mu} J$ a submersion. Since ϕ^{-1} is a fibre preserving diffeomorphism, $\tau | J^{-1}(\mu)$ is also a submersion, proving that it has constant rank $\dim Q^\mu$.

To verify the second statement, note $\mathrm{Ker}\, TJ_\alpha \supseteq T_\alpha(J^{-1}(\mu))$ always, so equality of these vector spaces holds if their dimensions are equal. From the last paragraph

$$\begin{aligned} \dim J^{-1}(\mu) &= \dim \mathrm{Ker}_{Q^\mu} J = \dim Q^\mu + \text{fibre dim } \mathrm{Ker}_{Q^\mu} J \\ &= \dim Q^\mu + n - (g-g_q) \end{aligned}$$

The calculation

$$\dim \mathrm{Ker}\, TJ_\alpha = 2n - (g-g_\alpha)$$

done earlier (§4, (iv)-(v) fact (b)) is valid here. These are equal, using the isotropy lemma, if (D1) holds. ■

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