Notes on a disturbing error found in the book Sub-Riemannian Geometry - general theory and examples' by Calin and Chang

The authors assert in Theorem 4.2.2., p 88, that a smooth curve ϕ is a minimizer to the fixed end pt problem (minimizing the usual action: integrated kinetic energy) if and only if

$$\dot{\phi} = \nabla S \qquad (*)$$

where S is a solution to the Hamilton-Jacobi [HJ] eqns and ∇S is the (sub)Riemannian gradient. The HJ equations are : $\frac{\partial S}{\partial t} = H(dS)$ where H is the (sub)Riemannian Hamiltonian, so that $H(dS) = \frac{1}{2}|\nabla S|^2$.

The error lies in the quantification. In their statement, and their proof, S is fixed once and for all, before ϕ.

To see that the assertion is wrong, suppose we are at a point p where $\nabla S \neq 0$ (where ∇ is their ∇_h in the subRiemannian setting and the Levi-Civita gradient in the Riemannian setting). This theorem would then imply that all local length minimizing geodesics through p POINT IN THE SAME DIRECTION. But this assertion is manifestly false, in the Euclidean plane, in any Riemannian manifold of dimension bigger than 1, and any subRiemannian manifold whose distribution has rank bigger than 1.

To see from another perspective how wrong this assertion is, take a solution to HJ for which $|\nabla S|$ is not constant along some geodesic ϕ. Then, for that geodesic the equation (*) cannot be satisfied because $\dot{\phi}$ does have constant length.

What the authors do prove, and prove well, is a modified version of 'only if' direction of the proof: If ϕ is a solution to (*) and if $\frac{\partial S}{\partial t} = const$ (the 'eikonal eqn) , then ϕ is a minimizing geodesic. (A bit of thought shows that along such a solution, we must have the Eikonal form of the HJ eqn: $c = \|\nabla S\|$.)

To see where their proof of the 'if' direction goes wrong, we work out a planar example. Take a geodesic $\phi(t)$ along the y-axis, and a soln to HJ whose gradient is in the x-direction. To get things to work easily, take a 'plane wave' solution to HJ, which is a sol'n of the form

$$S = -\theta(t/2) + \theta(ax + by)$$

with θ, a, b constants. A quick calculation shows that S is a soln to (*) if and only if $a^2 + b^2 = 1$. So, take

$$S = -t/2 + x$$

Then $\nabla S = e_1$. Take $\phi(t) = (0, t)$, $0 \leq t \leq 1$. The authors write down two fns

$$I(\phi) = (1/2) \int |\dot{\phi}|^2$$

and

$$J_S(\phi) = I(\phi) - \int dS$$

where the integral of dS is over the space-time path $(t, \phi(t))$ associated to ϕ. Thus

$$J_S = I(\phi) - (S(1, \phi(1)) - S(0, \phi(0))$$

so that for our S, and with our endpoints: $x(0) = x(1) = 0$ we have

$$J_S = I(\phi) + 1/2$$

Manifestly $J_S \geq 1/2$ for all paths ϕ on the unit interval satisfying $x(0) = x(1) = 0$. But, using their manipulations, which are all algebraically correct, they compute that

$$J = \int (1/2)|\dot{\phi} - \nabla S|^2 dt$$

Now the error occurs! They conclude that to minimize J we must have that $\dot{\phi} = \nabla S$. If this were true, we would have $J = 0$ along the path. But we have just computed that $J \geq 1/2$ for ANY path satisfying the endpoint conditions.

(Note that for our path $I = 1/2$ and $J = 1$.)

None of this has anything special to do with the sub-Riemannian world. The arguments can easily be 'lifted' to give subRiemannian counterexamples, for example, on the Heisenberg group.