Tubular Neighborhoods.

Consider an embedded submanifold X of a manifold M. The tubular neighborhood theorem asserts that sufficiently small neighborhoods of X in M are diffeomorphic to a vector bundle NX, called the "normal bundle of X", in such a way that under this diffeo X gets mapped to the zero section.

The utility of the tubular neighborhood theorem is very much like the utiility of the tangent space at a point. By using the tangent space at a point, we can reduce various questions in local analysis near that point to linear algebra. (Think of the statements of the IFTs.) Similarly, by using a tubular nbhd, we can reduce questions in analysis *near the submanifold* to analysis in the normal bundle which is *linear in the fiber*.

The normal bundle to $X \subset M$. First, if $E \subset F$ are vector bundles over X, then the quotient bundle F/E is a vector bundle with fibers F_x/E_x. If F is endowed with a fiberwise metric, then we can identify F/E with $E^{\perp}$.

Oops. What is a fiberwise metric ?? You fill in, or we fill in in class.

In the case of $X \subset M$ take for $E \subset F$ to be $TX \subset TM|_X$ where $TM|_X$ is just the ambient tangent bundle, restricted to points of X. In other words, the fiber of $F = TM|_X \to X$ at x is T_xM. (It does not have fibers over $y \notin X$.) Then the quotient bundle $E/F = TM|_X/TX$ is, by definition, the normal bundle to X.

A Riemannian metric on M induces fiber metrics. Then we have a geometric picture of the normal bundle to X as the set of vectors based at X, pointing *orthogonal* to X.

The rank of NX is the codimension of X in M.

Examples.

The normal bundle of $S^{n-1} \subset \mathbb{R}^n$ is isomorphic, as a smooth vector bundle, to the trivial rank one vector bundle $S^{n-1} \times \mathbb{R}$. More generally, if $X = \{f = 0\}$ is a hypersurface in $\mathbb{R}^n$ defined by the vanishing of a smooth function for which zero is a regular value, then $NX = X \times \mathbb{R}$, with the diffeo sending $(x, \lambda) \in X \times \mathbb{R}$ to $(x, \lambda \nabla f(x))$. We have used the basic fact from vector analysis that $\nabla f(x)$ is orthogonal to T_xX.

The normal bundle to $\mathbb{RP}^1 \subset \mathbb{RP}^2$ is the Mobius strip, viewed as *the* nontrivial rank 1 vector bundle over the circle (the circle being $\mathbb{RP}^1$.

Construction of normal bundle.

Introduce a Riemannian metric on M so that we can identify NX_x with $(T_xX)^{\perp}$. Now use the theory of geodesics. This theory, proved in detail in the Riemannian geometry course, asserts that through each point $(x, v) \in TM$ there is a unique curve γ called a geodesic, and playing the role of a straight line relative to the metric given, and satisfying the initial condition $\gamma(0) = x$, $\ddot{\gamma}(0) = v$. (If $M = \mathbb{R}^n$ with its standard metric, then the geodesics are the straight lines.)

Notation: $\gamma(t) = exp_x(tv)$.

We can restrict the exponential map to the normal bundle: $Exp_{NX} : NX \to M$ by sending $(x, n) \in NX$ to $exp_x(n)$. Under this map a line tn in the fiber $NX_x = T_xX^{\perp}$ gets sent to the unique geodesic tangent to n, and hence *orthogonal* to X, and passing through $x \in X$.

Theorem 1. *The map just constructed maps a neighborhood of the zero section in NX diffeomorphically onto a nbhd of X in M*

Proof. IFT.