Reading. Ch. 3 of Lee. Warner.

M is an abstract manifold. We have defined the tangent space to M via curves. We are going to give two other definitions. All three are used in the subject and one freely switches back and forth between the three.

Whatever the definition, the tangent space to M at a point $p \in M$ must enjoy the following properties. (1) it is a real vector space of dimension $n = dim(M)$

(2) it is intrinsically attached to M.

'Intrinsic' has two more-or-less equivalent meanings in differential geometry and topology: (a) coordinate independent,

(b): equivariant under mappings: if $F : M \to N$ is a smooth map between manifolds, then there should be defined, in a 'natural' way, a linear map $dF(p) : T_pM \to T_{F(p)}N$ between tangent spaces.

The three definitions are:

Definition 1. Via curves.

Definition 2. Via derivations acting on functions

Definition 3. Via coordinates.

Definition 1 is the one we have given already.

We recall it. Define an equivalence relation among the smooth curves passing through p. A tangent vector will be an equivalence class. This definition is the most 'geometric' of the three, but sometimes the most difficult to compute with.

Definition. A smooth curve c through p is a map $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$.

Definition. Two smooth curves $c, \tilde{c}$ through p "agree to first order" if, in some coordinate chart ϕ containing p their coordinate images agree to first order as curves in $\mathbb{R}^n$:

$$\phi \circ c(t) = \phi \circ \tilde{c}(t) + O(t^2)$$

Definition. [Curve def of tangent space.] A tangent vector at p is an equivalence class of curves passing through p. The tangent space is the set of all such equivalence classes of curves.

The following lemma shows that this definition is 'intrinsic' in the sense (a).

Lemma. Suppose $c, \tilde{c}$ are two curves through p which agree to 1st order in some chart ϕ. Then they agree to 1st order in any compatible chart.

Proof. Since $c, \tilde{c}$ agree to 1st order in the chart ϕ we have

$$\phi \circ c(t) = a + vt + O(t^2)$$
$$\phi \circ \tilde{c}(t) = a + vt + O(t^2)$$

where $a = \phi(p)$. Let ψ be another chart, and $g = \psi \circ \phi^{-1}$ Then $\psi \circ c = g\phi \circ c, \psi \circ \phi c(t) = g \circ \phi \circ \tilde{c}(t)$ So that

$$\phi \circ c(t) = g(a + vt + O(t^2)) = g(a) + [dg(a)v]t + O(t^2)$$
$$\phi \circ \tilde{c}(t) = g(a + vt + O(t^2)) = g(a) + [dg(a)v]t + O(t^2)$$

where we have used the fact that g is differentiable, and we have used the chain rule to Taylor expand out $g(a + vt + O(t^2))$. The two curves in the new coordinate system still agree to 1st order. QED

Definition 2 of the tangent space. Via derivations.

Write $C^\infty(M)$ for the space of all smooth functions on M. It is forms a commutative algebra over the reals. We can add, and multiply functions, and scalar multiply them.

Definiton. A derivation at p is an $\mathbb{R}$ linear map $C^\infty(M) \to \mathbb{R}$ which is a derivation in Leibnitz's sense:

$$v[fg] = f(p)v[g] + g(p)v[f]$$

In other words: it is a linear functional for which the product rule works.

Definition. The tangent space to M at p is the space of derivations at p.

That this definiton is intrinsic is clear. We did not use coordinates to define it. It is also obviously a vector space, since we can add derivations and multiply them by real numbers. But the dimension of this vector space is not obvious. Is it finite? Maybe there are too many functions? Or is it zero? How do we know a manifold has ANY globally defined functions?

Exercise. If f and g agree in a nbhd of p and v is a derivation at p, then $v[f] = v[g]$.

Prop. If f is smooth defined in a coord nbhd of p, then it can be smoothly extended to all of M.

Pf. Bump functions.

Cor. The value of $v[f]$ is well-defined when f is defined on a nbhd of p, rather than all of M.

(Rmk. Smooth Urysohn lemma.)

Cor. to exercise. Let U be any nbhod of p and consider the space $C^\infty(U)$ of smooth functions on To answer this last question will lead us to bump functions, an important tool later on.

Definition 3 of the tangent space. via coordinates.

Consider all charts $\psi_\alpha : U_\alpha \subset M \to \mathbb{R}^n$ containing p, so that $p \in U_\alpha$. For each such chart introduce a copy of $\mathbb{R}^n$, denoted by $\mathbb{R}^n_\alpha$, and form the disjoint union of all these vector spaces. :

$$\coprod_{\{\alpha : p \in U_\alpha\}} \mathbb{R}^n_\alpha$$

. The funny symbol $\coprod$ indicates DISJOINT union. Define an equivalence relation $\sim$ on this disjoint union by declaring $v_\alpha \in \mathbb{R}^n_\alpha$, and $v_\beta \in \mathbb{R}^n_\beta$ to be equivalent: $v_\alpha \sim v_\beta$: iff $d(\psi_\beta \circ \psi_\alpha^{-1})(\psi_\alpha(p))v_\alpha = v_\beta$. Then define $T_pM = \coprod_{\{\alpha : p \in U_\alpha\}} \mathbb{R}^n_\alpha / \sim$.

This definition is clearly intrinsic according to notion (a) of 'intrinsic', since we have divided out by all choice of charts. It is also clearly a vector space of dimension n. Fix any one chart $\psi_\alpha : U_\alpha \subset M - \to \mathbb{R}^n$. Then every $[v] \in T_pM$ has a unique representative $v_\alpha \in \mathbb{R}^n_\alpha$ so that $\mathbb{R}^N_\alpha \cong T_pM$.

Pros and cons. This is the definition closest to the way computations are done. It is the ugliest definition.

Bases in the various definitions. Standard notation. Equivalences between defintions.

Let x^i be coordinates defined in a nbhd of p. Thus, some coordinate chart $\psi : U \subset M \to \mathbb{R}^n$ is written out as $\psi(q) = (x^1(q), \ldots, x^n(q))$. In all three definitions, these coordinates define a basis $\frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$ for T_pM. Thus a typical element of T_pM can be uniquely expressed as $\Sigma v^i \frac{\partial}{\partial x^i}$ with $v^i \in \mathbb{R}$.

Basis in definition 1. Consider the coordinate x^i-curve γ_i. By this we mean the curve in $\mathbb{R}^n$ parallel to the x^i axis and through the point corresponding to p: thus:$\gamma_i(t)$ is defined by $x^j(t) = x_0^j$, $x^i(t) = x_0^i + t$, where x_0^i are the coordinates of p. Then $\frac{\partial}{\partial x^i}$ means the equivalence class of the curve $\psi^{-1}(\gamma_i)$.

Basis in definiton 2. If $f \in C^\infty(M)$ then $f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}$ can be written (by a slight abuse of notation) as $f(x^1, \ldots, x^n)$. The derivation $\frac{\partial}{\partial x^i}$ is the usual partial derivative: $\frac{\partial}{\partial x^i}[f] = \frac{\partial f}{\partial x^i}$

Basis in definition 3. $\frac{\partial}{\partial x^i}$ is the vector whose representative in the chart for $\psi = \psi_\alpha$, is the vector $e_i = (0, \ldots 0, 1, 0, \ldots) \in \mathbb{R}^n_\alpha$.

Equivalences of definitons.

Def 1 to 2. To define a derivation v from a curve c we set $v[f] = (\frac{d}{dt}|_{t=0}(f(c(t)))$. Note: $\mathbb{R} \xrightarrow{c} M \xrightarrow{f} \mathbb{R}$

Coordinate computations shows $c \mapsto v$ is well-defined on the level of equivalence classes: if $c_1 \sim c_2$ then $(\frac{d}{dt}|_{t=0}(f(c_1(t)) = (\frac{d}{dt}|_{t=0}(f(c_1(t))$. A coordinate computation also shows that the map $c'(0) \to v$ is linear. If $v = \Sigma v^i \frac{\partial}{\partial x^i}$ we compute, using the chain rule that indeed:

$$v[f] = \Sigma v^i \frac{\partial}{\partial x^i}[f]$$

as it should.

Definition 2 corresponds to the directional derivative of vector calculus.

Def 1 to Def 3. We went from 1 to 3 when we showed that the equivalence class of def 1 was chart-independent. (The lemma following the explanation of def 1: write the curve in a coord chart. Its first order derivative is the tangent vector as represented in that chart.) Conversely, given a chart $\phi_\alpha : U_\alpha \to \mathbb{R}^n = \mathbb{R}^n_\alpha$ with P_0 corresponding to p, the inverse image under ϕ_α of the curve $P_0 + tv \in \mathbb{R}^n_\alpha$ is the curve whose equivalence class represents v.

Def 3 to 2. The vector $v = (v^1, \ldots v^n) \in \mathbb{R}^n_\alpha$ represents the derivation $\Sigma v^i \frac{\partial}{\partial x^i}$.

1.

More on Def 2. Why is the space of derivations based at p spanned by the $\frac{\partial}{\partial x^i}$? It is not even clear that this space of derivations is finite-dimensional.

Define bump functions. Discuss bump fns. Extending fns. Why we can think of x^i as fns on all of M.

Impossibility for cx mfds.

2. Derivations. Maximal ideals. Cotangent space.

We state some algebraic consequences of the definition. Let v be a derivation at p, and let $\mathfrak{m}_p \subset C^\infty(M)$ be the subalgebra of functions vanishing at p.

Fact 1. $v[c] = 0$, c a constant function.

Proof: $1 = 1 \cdot 1$, so $v[1] = 1v[1] + 1v[1] = 2v[1]$, implying that $v[1] = 0$. Now $v[c] = cv[1]$.

Consequence. $v[f] = v[f - f(p)]$.

3.

Fact 2. If $h = \mathfrak{m}_p^2$ then $v[h] = 0$.

Proof $h \in \mathfrak{m}_p^2$ means $h = fg$ for some $f, g \in \mathfrak{m}_p$. Then $v[h] = f(p)v[g] + g(p)v[f] = 0 + 0$ since $f(p) = g(p) = 0$.

Proposition 0.1. *v defines a linear map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}$ and is uniquely determined by this map.*

Proof: By fact 1 the value of v is determined by its values on $\mathfrak{m}_p$. Any linear map $L : E \to F$ between vector spaces descends canonically to a 'quotient map" $E/ker(L) \to F$, and, if we know $ker(L)$, then L is uniquely determined by this quotient map.

3.

Fact 3. $\mathfrak{m}_p/\mathfrak{m}_p^2$ is a finite-dimensional vector space of dimension n.

Proof of Fact 3. The n coordinates are the 1st order Taylor expansion of a function. In detail, use coordinates x^i, $i = 1, \ldots, n$ centered at p. NOTE: "Centered at p" means that $x^i(p) = 0$.) Then Fact 2 transfers to the same statement regarding $\mathfrak{m}_0 \subset C^\infty(\mathbb{R}^n)$. Expand f in terms of the x^i: $f = \Sigma a_i x^i + O(|x|^2)$. We must show that the remainder term $O(|x|^2)$ is in $\mathfrak{m}_p$. This is easily seen from

Lemma 0.2 (Hadamard's lemma.)**.** *Let f be a smooth function vanishing at $0 \in \mathbb{R}^n$. Then we can write $f = \Sigma x^i g_i(x_1, \ldots, x_n)$ where the g_i are smooth functions satisfying $g(0) = \frac{\partial f}{\partial x^i}|_0$*

Proof of Hadamard. Write $f(x) = f(0) + \int \frac{d}{dt} f(tx)dt$ Use $f(0) = 0$ and expand out the integral to obtain $f(x) = \Sigma x^i g_i(x)$ where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(tx)dt$. QED (also see, Lee, appendix).

End of proof of Fact 3. Hadamard's lemma asserts that by subtracting off the linear Taylor expansion, $\Sigma a_i x^i$ from f we obtain a function in $\mathfrak{m}_0^2$.

Definition. $\mathfrak{m}_p/\mathfrak{m}_p^2$ is called the "space of covectors" at p or the cotangent bundle at p and is denoted by T_p^*M.

Then the proposition asserts that the tangent space T_pM (viewed as derivations) is canonically dual to T_p^*M.

Discussion.

There is a long tradition of recovering and better understanding a manifold, variety, topological space,... by way of algebras of functiosn on it.

the correspondence $p \mapsto \mathfrak{m}_p \subset C^\infty(M)$ embeds M as a family of maximal ideals within $C^\infty(M)$.

Perhaps the most striking is in the subject of C^* -algebras , the theorem called the Gelfand-Naimark theorem. Let M be a compact Hausdorff space and $\mathfrak{A}$ the space $C^0(M, \mathbb{C})$ of continous complex-valued functions on M, It is a commutative algebra over $\mathbb{C}$. The sup norm, and the operation $f \mapsto \bar{f}$, gives $\mathfrak{A}$ the structure of what is known as a C^*-algebra.

Gelfand-Naimark Theorem. Every commutative C^* algebra is $C^0(M, \mathbb{C})$ for some compact Hausdorff space M.

The space M in the Gelfand-Naimark theorem is built out of "multiplicative linear functionals" and forms the "spectrum" of $\mathfrak{A}$. Conversely, given M, the associated "multiplicative linear functional" is evaluation at p: $f \mapsto f(p)$, a map from $\mathfrak{A} \to \mathbb{C}$. The kernel of this map corresponds to our $\mathfrak{m}_p$.

Algebraic geometry. The subject begins by studying the zero locuses of polynomial functions. These zero locuses are called "varieties". Varieties are very often manifolds, and are always manifolds at most of their points – like our cone or cross. An essential theme in algebraic geometry is the traffic back and forth between the variety and the algebra of polynomial functions defined on the variety. The variety is reconstructed out of the algebra, as the space of maximal ideals in the algebra. Our definition of the cotangent space and tangent space work in the algebraic setting, even over arbitrary rings (!) and are called the Zariski (co)tangent space.